# Further Insights into SCCs

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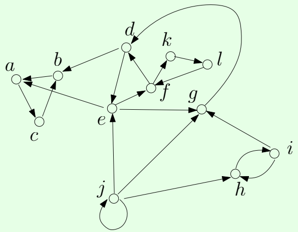
Given a directed graph $G = (V, E)$, the goal of the **strongly connected components problem** is to divide $V$ into disjoint subsets, each being an SCC.

**Example:**




We should output: $\{a, b, c\}$, $\{d, e, f, g, k, l\}$, $\{h, i\}$, and $\{j\}$.

## Algorithm

**Step 1:** Run DFS on $G$ and list the vertices by the order they turn black.

- If a vertex is the $i$-th vertex turning black, define its **label** as $i$.

**Step 2:** Obtain the **reverse graph** $G^{rev}$ by flipping all the edge directions in $G$.

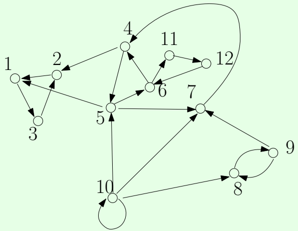
**Step 3:** Perform DFS on $G^{rev}$ subject to the following rules:

- **Rule 1:** Start at the vertex with the largest label.
- **Rule 2:** When a restart is needed, do so from the white vertex with the largest label.

Output the vertices in each DFS-tree as an SCC.

Next, we will show how to implement the SCC algorithm in $O(|V| + |E|)$ time. You can assume that $V = \{1, 2, ..., n\}$.

**Example:**

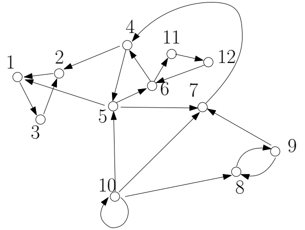



$G$

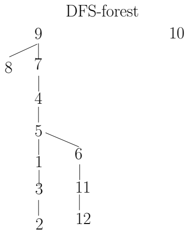
## Step 1

Perform DFS on $G$ and record the turn-black order in an array $A$.

- $A[i]$ stores the vertex with label $i$.




$G$




vertex with label 1 — vertex with label 2

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 8 | 2 | 3 | 1 | 12 | 11 | 6 | 5 | 4 | 7 | 9 | 10 |

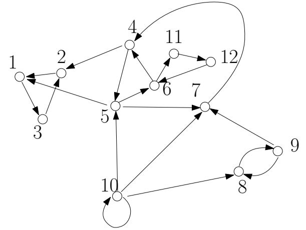
$A$

Time: $O(|V| + |E|)$.

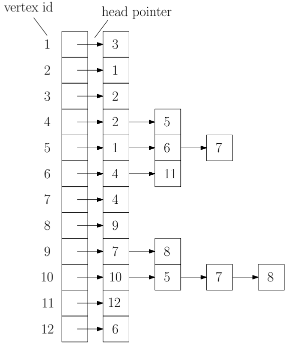
Step 2

Obtain $G^{rev} = (V, E^{rev})$ from $G$ in $O(|V| + |E|)$ time.

We will illustrate how to do so through an example.



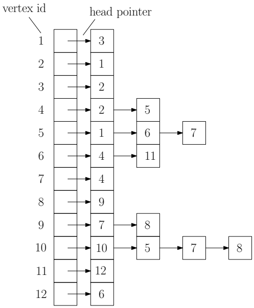

$G$



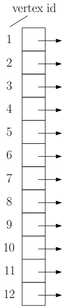

adjacency list of $G$

## Step 2

Initialize the head-pointer array for $G^{rev}$.



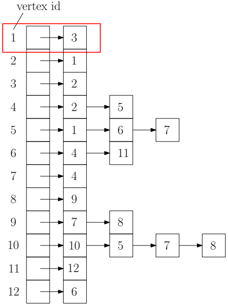

adj. list of $G$



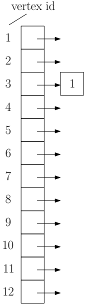

adj. list of $G^{rev}$

## Step 2

Scan the neighbor list of each $u \in V$ in $G$.
For every out-neighbor $v$ of $u$, add $u$ to the neighbor list of $v$ in $G^{rev}$.




adj. list of $G$




adj. list of $G^{rev}$

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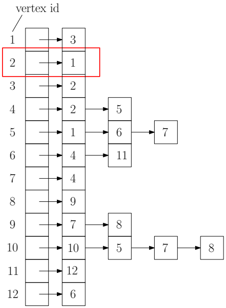
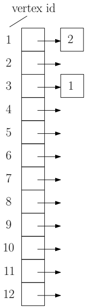



adj. list of $G$




adj. list of $G^{rev}$

## Step 2

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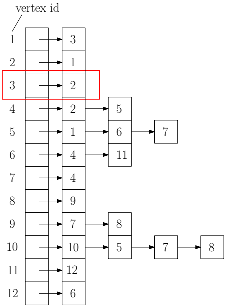
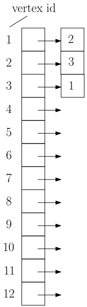
For every out-neighbor $v$ of $u$, add $u$ to the neighbor list of $v$ in $G^{rev}$.



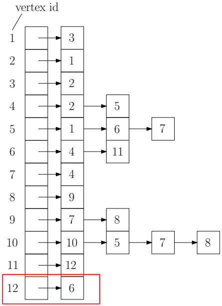

adj. list of $G$



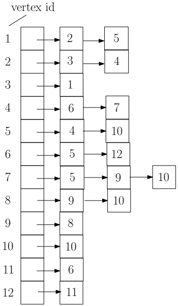

adj. list of $G^{rev}$

## Step 2

Scan the neighbor list of each $u \in V$ in $G$.
For every out-neighbor $v$ of $u$, add $u$ to the neighbor list of $v$ in $G^{rev}$.




adj. list of $G$




adj. list of $G^{rev}$

## Step 3

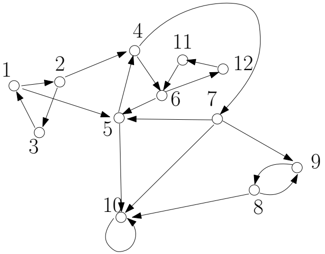
Perform DFS on $G^{rev}$ and use $A$ to select the vertex to start/restart from.

the 1st starting vertex

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 8 | 2 | 3 | 1 | 12 | 11 | 6 | 5 | 4 | 7 | 9 | 10 |

vertex with label 12

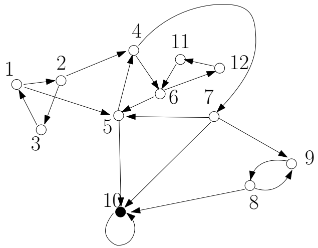
$A$




$G^{rev}$

## Step 3

Start the 1st DFS on $G^{rev}$ from vertex 10. Output $\{10\}$.




$G^{rev}$

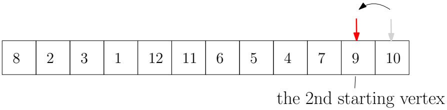
DFS-tree

10

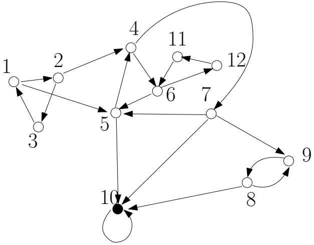
Vertex 10 is now black.

## Step 3

Scan $A$ backwards from $A[12]$ and find the first white vertex $A[11] = 9$.



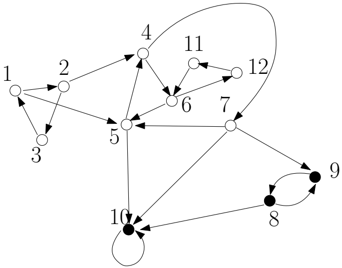

$A$



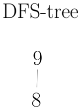

$G^{rev}$

## Step 3

Start the 2rd DFS on $G^{rev}$ from 9. Output $\{8, 9\}$.



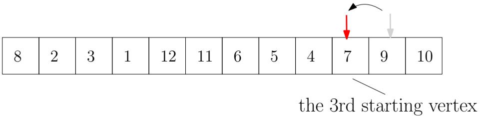

$G^{rev}$



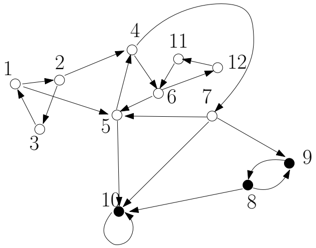

Vertices 8 and 9 are now black.

## Step 3

Scan $A$ backwards from $A[11]$ and find the first white vertex $A[10] = 7$.



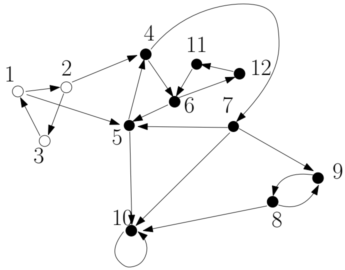

$A$




$G^{rev}$

## Step 3

Start the 3rd DFS on $G^{rev}$ from 7. Output $\{7, 5, 4, 6, 12, 11\}$.



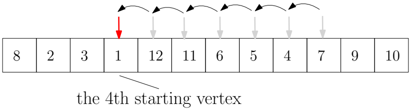

$G^{rev}$

DFS-tree

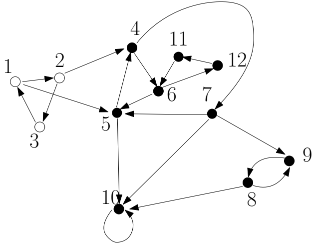

Vertices 7, 5, 4, 6, 12, and 11 are now black.

## Step 3

Scan $A$ backwards from $A[10]$ and find the first white vertex $A[4] = 1$.



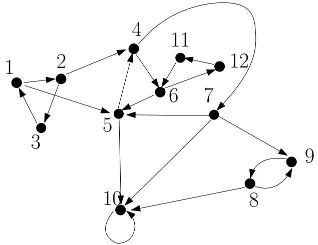

$A$



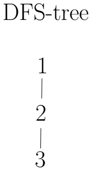

$G^{rev}$

## Step 3

Start the 4th DFS on $G^{rev}$ from 1. Output $\{1, 2, 3\}$.




$G^{rev}$

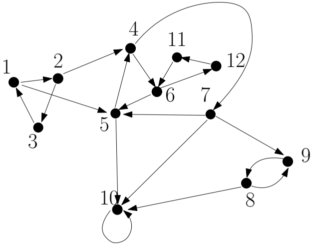


## Step 3

Scan $A$ backwards from 1 and find no other white vertices.
The algorithm finishes.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 8 | 2 | 3 | 1 | 12 | 11 | 6 | 5 | 4 | 7 | 9 | 10 |

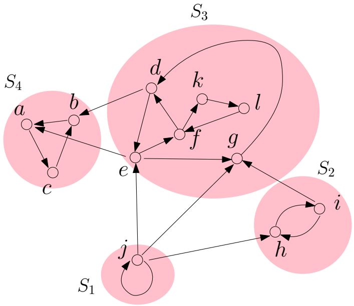
$A$



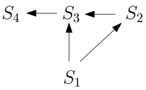

$G^{rev}$

Next, we will unveil a mathematical structure of the SCC problem that suggests a generic algorithmic paradigm.

## Sink SCC




SCC Graph $G^{scc}$



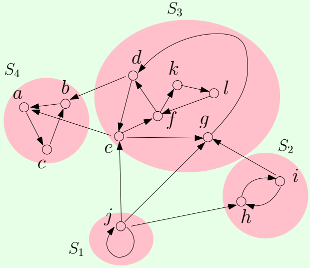

An SCC is a **sink SCC** if it has no outgoing edge in $G^{scc}$.

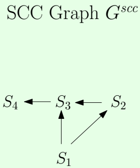
$S_4$ is the only sink SCC in the above example.

# A conceptual SCC strategy

1. **while** $G^{scc}$ not empty **do**
2. $S \leftarrow$ a sink SCC
3. run DFS from any vertex in $S$
4. remove all the vertices in $S$ from $G$; delete vertex $S$ from $G^{scc}$

**Example:**



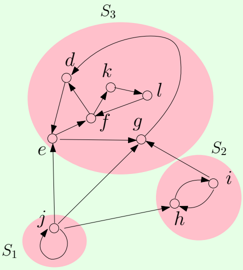



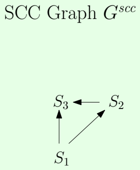

DFS from anywhere in $S_4$ finds SCC $\{a, b, c\}$.

## A conceptual SCC strategy

1. **while** $G^{scc}$ not empty **do**
2. $S \leftarrow$ a sink SCC
3. run DFS from any vertex in $S$
4. remove all the vertices in $S$ from $G$; delete vertex $S$ from $G^{scc}$

### Example:



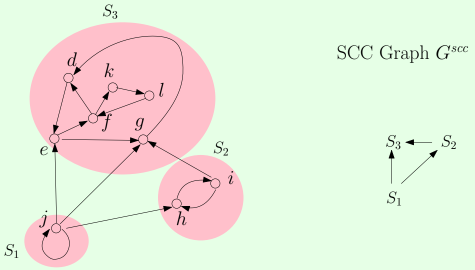




Delete $S_4$ from $G$ and $G^{scc}$. New sink vertex: $S_3$.

## A conceptual SCC strategy

1. **while** $G^{scc}$ not empty **do**
2. $\quad$ $S \leftarrow$ a sink SCC
3. $\quad$ run DFS from any vertex in $S$
4. $\quad$ remove all the vertices in $S$ from $G$;
   $\quad$ delete vertex $S$ from $G^{scc}$

**Example:**

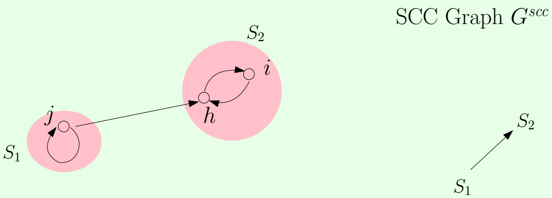



DFS from anywhere in $S_3$ finds SCC $\{d, e, f, g, k, l\}$.

## A conceptual SCC strategy

1. **while** $G^{scc}$ not empty **do**
2. $\quad$ $S \leftarrow$ a sink SCC
3. $\quad$ run DFS from any vertex in $S$
4. $\quad$ remove all the vertices in $S$ from $G$;
   delete vertex $S$ from $G^{scc}$

**Example:**

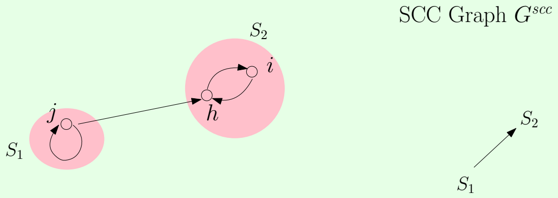



Delete $S_3$. New sink vertex: $S_2$.

## A conceptual SCC strategy

1. **while** $G^{scc}$ not empty **do**
2. $S \leftarrow$ a sink SCC
3. run DFS from any vertex in $S$
4. remove all the vertices in $S$ from $G$; delete vertex $S$ from $G^{scc}$

**Example:**




DFS from anywhere in $S_2$ finds SCC $\{i, h\}$.

## A conceptual SCC strategy

1. **while** $G^{scc}$ not empty **do**
2. $\quad$ $S \leftarrow$ a sink SCC
3. $\quad$ run DFS from any vertex in $S$
4. $\quad$ remove all the vertices in $S$ from $G$;
   delete vertex $S$ from $G^{scc}$

**Example:**


Delete $S_2$. New sink vertex $S_1$.

## A conceptual SCC strategy

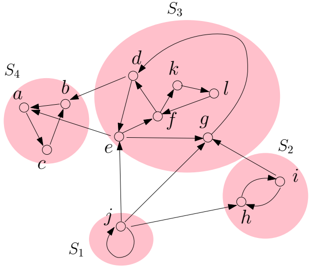
1. **while** $G^{scc}$ not empty **do**
2. $S \leftarrow$ a sink SCC
3. run DFS from any vertex in $S$
4. remove all the vertices in $S$ from $G$; delete vertex $S$ from $G^{scc}$

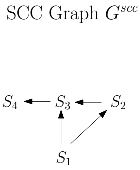
**Example:**


The 4th DFS finds SCC $\{j\}$.

**Question:**
Why does our SCC algorithm work on the **reverse** graph, as opposed to the **original** one?

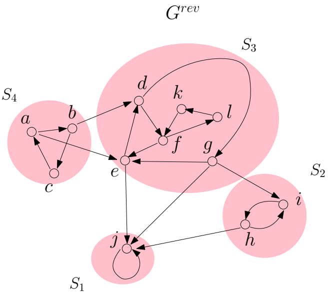
**Answer:** Non-trivial to find the next sink SCC.

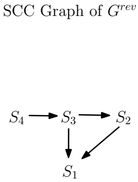



**Not easy:** You need to find a vertex in $S_4$ first, then a vertex in $S_3$, then one in $S_2$, and finally in $S_1$.

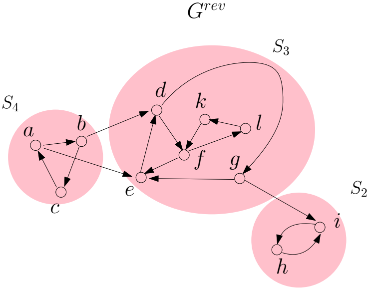
It turns out that finding the next sink SCC on the **reverse** graph is **much** easier.

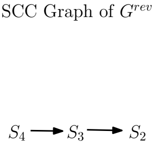



Sink SCC = $S_1$.
DFS from $j$ finds SCC $\{j\}$

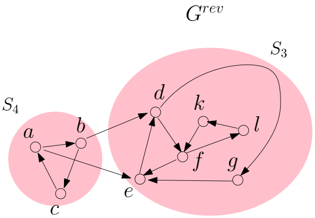
It turns out that finding the next sink SCC on the reverse graph is **much** easier.

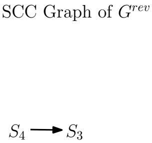



Sink SCC $= S_2$.
DFS from anywhere in $S_2$ finds SCC $\{h, i\}$

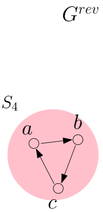
It turns out that finding the next sink SCC on the reverse graph is **much** easier.

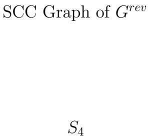



Sink SCC $= S_3$.
DFS from anywhere in $S_3$ finds SCC $\{d, e, f, g, k, l\}$.

It turns out that finding the next sink SCC on the reverse graph is **much** easier.




Sink SCC $= S_4$. The last DFS finds SCC $\{a, b, c\}$.

This is exactly how our SCC algorithm finds the SCCs.