Dynamic Programming: Evaluating Recursive Functions

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Pitfall of Recursion

A recursive algorithm does considerable redundant work if the same subproblem is encountered over and over again.

Problem 1

Let A be an array of n integers. Define a function $f(x)$ — where $x \geq 0$ is an integer — as follows:

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \max_{i=1}^{x}(A[i] + f(x-i)) & \text{otherwise} \end{cases}$$

Consider the following algorithm for calculating $f(x)$:

```
algorithm $f(x)$
  if $x = 0$ then return 0
  $max = -\infty$
  for $i = 1$ to $x$
      $v = A[i] + f(x - i)$
      if $v > max$ then $max = v$
  return $max$
```

Prove: The above algorithm takes $\Omega(2^n)$ time to calculate $f(n)$.

Solution

We will prove the statement by induction. Executing $f(n)$ will launch function calls $f(n-1), f(n-2), \cdots, f(0)$.

Let $g(n)$ denote the running time of $f(n)$. So we have:

$$\begin{aligned} g(0) &\geq 1, \\ g(1) &\geq 1, \\ g(n) &\geq \sum_{i=0}^{n-1} g(i) \text{ for } n \geq 2. \end{aligned}$$

We will prove $g(n) \geq 2^{n-1}$ for all $n \geq 1$ by induction on n.

Solution

The base case $n = 1$ is obviously correct. Next, assuming $g(n) \geq 2^{n-1}$ for $n \leq k$ where k is an integer at least 1, we will prove $g(k+1) \geq 2^k$.

As $k + 1 \geq 2$, we have:

$$g(k+1) \quad \geq \quad \sum_{i=0}^{k} g(k).$$

By the inductive assumption, we have:

$$g(k+1) \quad \geq \quad 1 + \sum_{i=1}^{k} 2^{k-1} = 2^k.$$

□

Principle of Dynamic Programming

Resolve subproblems according to a certain order. Remember the output of every subproblem to avoid re-computation.

Problem 2

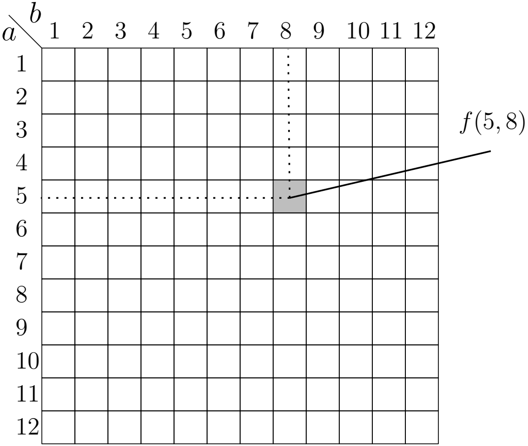
Let A be an array of n integers. Define function $f(a, b)$ — where $a \in [1, n]$ and $b \in [1, n]$ — as follows:

$$f(a,b) = \begin{cases} 0 & \text{if } a \geq b \\ \left(\sum_{c=a}^{b} A[c]\right) + \min_{c=a+1}^{b-1}\{f(a,c) + f(c,b)\} & \text{otherwise} \end{cases}$$

Design an algorithm to calculate $f(1, n)$ in $O(n^3)$ time.

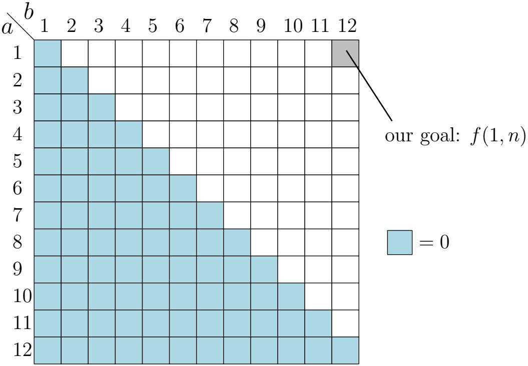
Solution

List all the subproblems.



Solution

$f(a, b) = 0$ when $a \geq b$.



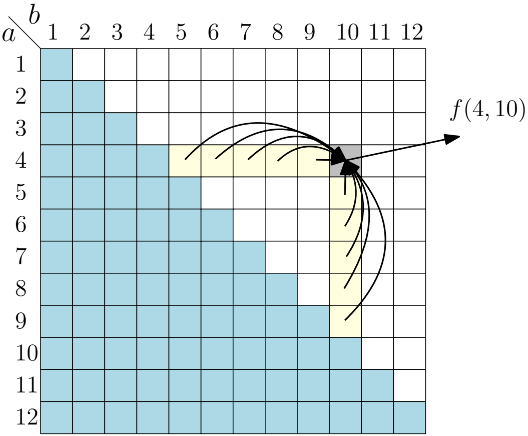
our goal: $f(1, n)$

🟦 $= 0$

Solution

$$f(a,b) = \left(\sum_{c=a}^{b} A[c]\right) + \min_{c=a+1}^{b-1}\{f(a,c) + f(c,b)\} \text{ when } a < b.$$

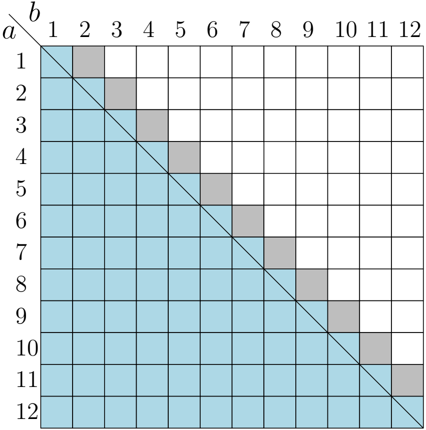
Find out the dependency relationships.



Solution

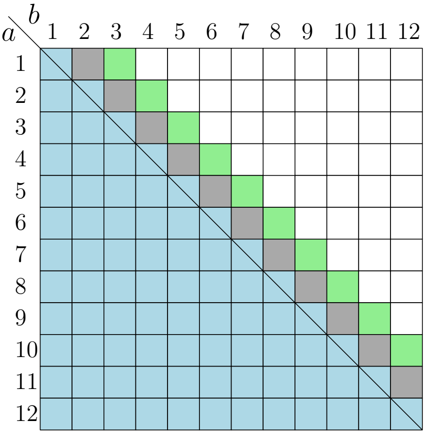
$f(a, b) = (\sum_{c=a}^{b} A[c]) + \min_{c=a+1}^{b-1}\{f(a, c) + f(c, b)\}$ when $a < b$.

Let us start with the gray cells — they correspond to $f(a, b)$ where $a = b - 1$. These cells depend on no other cells.



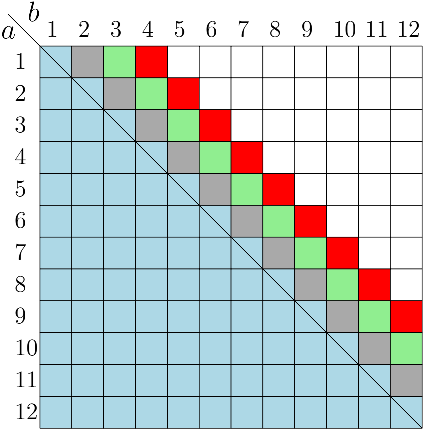
Solution

Let us continue with the green cells — they correspond to $f(a, b)$ where $a = b - 2$. Every such cell depends on two gray cells, which have already been computed.



Solution

Let us continue with the red cells — they correspond to $f(a, b)$ where $a = b - 3$. Every such cell depends on two gray cells and two green cells, all of which have been computed.



$a \backslash b$	1	2	3	4	5	6	7	8	9	10	11	12
1	🟦	▨	🟩	🟥								
2	🟦	🟦	▨	🟩	🟥							
3	🟦	🟦	🟦	▨	🟩	🟥						
4	🟦	🟦	🟦	🟦	▨	🟩	🟥					
5	🟦	🟦	🟦	🟦	🟦	▨	🟩	🟥				
6	🟦	🟦	🟦	🟦	🟦	🟦	▨	🟩	🟥			
7	🟦	🟦	🟦	🟦	🟦	🟦	🟦	▨	🟩	🟥		
8	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	▨	🟩	🟥	
9	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	▨	🟩	🟥
10	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	▨	🟩
11	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	▨
12	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦

Solution

The order can be summarized as follows.

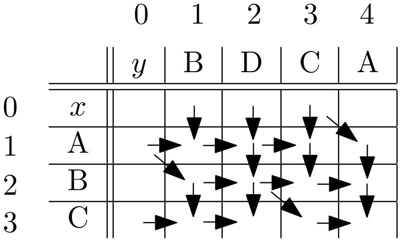
- All cells $f(a,b)$ with $b-a=1$, each computed in $O(1)$ time.
- All cells $f(a,b)$ with $b-a=2$, each computed in $O(2)$ time.
- ...
- All cells $f(a,b)$ with $b-a=k$, each computed in $O(k)$ time.
- ...
- All cells $f(a,b)$ with $b-a=n-1$, each computed in $O(n-1)$ time.

There are $O(n^2)$ values to calculate.
Total time complexity $= O(n^3)$.

Problem 3 (Space Consumption)

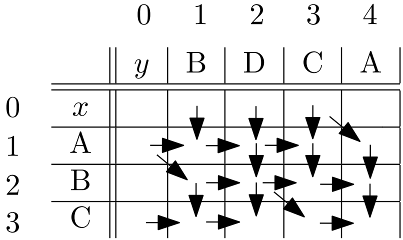
In Lecture Notes 8, our algorithm for computing $f(n, m)$ used $O(nm)$ space. Next, we will reduce the space complexity to $O(n + m)$.

Recall the dependency graph:



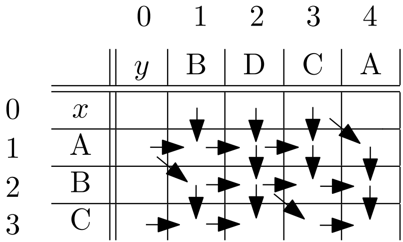
Solution

We can calculate the values in the row-major order, i.e., row 0 to row 3 and left to right in each row. We used $O(mn)$ space because we stored all the values. Observe, however, that only two rows need to be stored at any moment .



Solution

Same idea for the column-major order.



So the space complexity is $O(\min\{m, n\})$, in addition to the $O(n + m)$ space needed to store x and y.

Think: Can this trick be used to reduce the space in Problem 2?