Tutorial 12: Further Discussion on Set Cover and Hitting Set

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Set Cover

Let U be a finite set called the **universe**.

We are given a family $\mathcal{S}$ where

- each member of $\mathcal{S}$ is a set $S \subseteq U$;
- $\bigcup_{S \in \mathcal{S}} S = U$.

A sub-family $\mathcal{C} \subseteq \mathcal{S}$ is a **universe cover** if every element of U appears in at least one set in $\mathcal{C}$.

- Define the **cost** of $\mathcal{C}$ as $|\mathcal{C}|$.

The set cover problem:
Find a universe cover with the smallest cost.

Example: $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $\mathcal{S} = \{S_1, S_2, ..., S_5\}$ where

$$\begin{aligned} S_1 &= \{1, 2, 3, 4\} \\ S_2 &= \{2, 5, 7\} \\ S_3 &= \{6, 7\} \\ S_4 &= \{1, 8\} \\ S_5 &= \{1, 2, 3, 8\}. \end{aligned}$$

An optimal solution is $\mathcal{C} = \{S_1, S_2, S_3, S_4\}$.

Our Approximation Algorithm

$\mathcal{C} = \emptyset$
while U still has elements not covered by any set in $\mathcal{C}$
    $F \leftarrow$ the set of elements in U not covered by any set in $\mathcal{C}$
    /* for each set $S \in \mathcal{S}$, define its **benefit** to be $|S \cap F|$ */
    add to $\mathcal{C}$ a set in $\mathcal{S}$ with the largest benefit
return $\mathcal{C}$

We proved in the lecture that the algorithm is $(1 + \ln |U|)$-approximate.

Next, we will prove that the algorithm is also h-approximate, where $h = \max_{S \in \mathcal{S}} |S|$.

Example: $\mathcal{S} = \{S_1, S_2, ..., S_5\}$ where

$$\begin{aligned} S_1 &= \{1,2,3,4\} \\ S_2 &= \{2,5,7\} \\ S_3 &= \{6,7\} \\ S_4 &= \{1,8\} \\ S_5 &= \{1,2,3,8\}. \end{aligned}$$

Then, $h = 4$.

Theorem: The algorithm returns a universe cover with cost at most $h \cdot OPT_{\mathcal{S}}$.

Proof. Suppose that our algorithm picks t sets. Every time the algorithm picks a set, at least one **new** element is covered. For each $i \in [1, t]$, denote by e_i an arbitrary element that is **newly** covered when the i-th set is picked.

Let $\mathcal{C}^*$ be an optimal universe cover. Because each e_i exists in at least one set of $\mathcal{C}^*$, we have:

$$\begin{aligned} t = \sum_{i=1}^{t} 1 &\leq \sum_{i=1}^{t} \#\text{ sets in } \mathcal{C}^* \text{ containing } e_i \\ &\leq \sum_{e \in U} \#\text{ sets in } \mathcal{C}^* \text{ containing } e \\ &= \sum_{S \in \mathcal{C}^*} |S| \leq |\mathcal{C}^*| \cdot h. \end{aligned}$$

Corollary: If $h = O(1)$, then our algorithm achieves a constant approximation ratio.

Remark: With a more careful analysis, we can actually prove that our algorithm has an approximation ratio of $1 + \ln h$.

- Not required in this course.

Our set cover algorithm can be used to solve many problems with approximation guarantees. Next, we will see two examples.

Vertex Cover

$G = (V, E)$ is an undirected graph. We want to find a small subset $V' \subseteq V$ such that every edge of E is incident to at least one vertex in V'. The optimization goal is to minimize $|V'|$.

Convert the problem to set cover:

- For every $v \in V$, define S_v = the set of edges incident on v.
- Apply our algorithm on the set-cover instance: $\mathcal{S} = \{S_v \mid v \in V\}$.

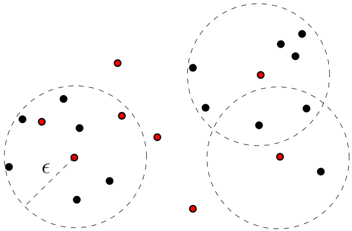
This gives an $O(\ln |V|)$-approximate solution.

Remark: We have already learned how to ensure an approximation ratio of 2. But the point here is to demonstrate the usefulness of set cover, rather than improving the approximation ratio.

Red-Black Coverage

R = a set of n red points in 2D space
B = a set of n black points in 2D space
ϵ = a positive integer.

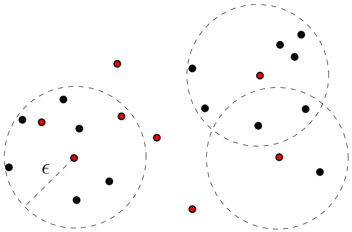
A subset $S \subseteq R$ is a B-**guarding set** if, for every black point $b \in B$, there is at least one point $r \in S$ with $dist(r, b) \le \epsilon$.



OPT = the smallest size of all B-guarding sets.

Goal: Return a B-guarding set with size $OPT \cdot O(\log n)$ (assume that at least one B-guarding set exists).

Red-Black Coverage



Convert the problem to set cover:

- For every $r \in R$, define S_r = the set of black points b satisfying $dist(r, b) \le \epsilon$.
- Apply our algorithm on the set-cover instance: $\mathcal{S} = \{S_r \mid r \in R\}$.

This gives an $O(\log n)$-approximate solution.

Next, we will turn our attention to the hitting set problem, which is in fact equivalent to set cover.

Hitting Set

Let U be a finite set called the **universe**.

We are given a family $\mathcal{S}$ where

- each member of $\mathcal{S}$ is a set $S \subseteq U$;
- $\bigcup_{S \in \mathcal{S}} S = U$.

A subset $H \subseteq U$ **hits** a set $S \in \mathcal{S}$ if $H \cap S \neq \emptyset$.
A subset $H \subseteq U$ is a **hitting set** if it hits all the sets in $\mathcal{S}$.

The hitting set problem:
Find a hitting set H of the minimize size.

Example: $U = \{1, 2, 3, 4, 5\}$ and $\mathcal{S} = \{S_1, S_2, ..., S_8\}$ where

$$\begin{aligned} S_1 &= \{1, 4, 5\} \\ S_2 &= \{1, 2, 5\} \\ S_3 &= \{1, 5\} \\ S_4 &= \{1\} \\ S_5 &= \{2\} \\ S_6 &= \{3\} \\ S_7 &= \{2, 3\} \\ S_8 &= \{4, 5\} \end{aligned}$$

An optimal solution is $H = \{1, 2, 3, 4\}$.

Next, we will provide a matrix-view of set cover and hitting set, which hopefully will help you better understand their equivalence. We will achieve the purpose through a "bridging problem" defined on a matrix.

$\boldsymbol{M}$ = an $n \times m$ matrix.
$M[i, j] = 0$ or 1 for every $i \in [1, n]$ and $j \in [1, m]$.
Constraint: At least one 1 at each row and at each column.
Row Cover: a set R of rows s.t. every column has at least one 1 at the rows of R.
OPT_{row} : the minimum size of all row covers.

Example

$$\boldsymbol{M} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

An optimal row cover takes the first four rows.

Using our set-cover algorithm, we can find a row cover of size $OPT_{row} \cdot O(\log m)$.

Let us now relate the matrix problem to hitting set.

Consider the hitting set instance $U = \{1, 2, 3, 4, 5\}$ and $\mathcal{S} = \{S_1, S_2, ..., S_8\}$ where $S_1 = \{1, 4, 5\}$, $S_2 = \{1, 2, 5\}$, $S_3 = \{1, 5\}$, $S_4 = \{1\}$, $S_5 = \{2\}$, $S_6 = \{3\}$, $S_7 = \{2, 3\}$, and $S_8 = \{4, 5\}$.

We can describe the instance with

$$\boldsymbol{M} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where the i-th row corresponds to integer $i \in U$ and the j-th column corresponds to S_j. Now, the goal is to find an optimal row cover! We can find an $O(\log m)$ approximation using our set-cover algorithm.

We have seen why hitting set can be converted to set cover. We will now discuss the opposite.

Consider the matrix row cover problem again.

Example

$$\boldsymbol{M} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

An optimal row cover takes the first four rows.

We can also interpret the problem as a **hitting set** problem!

See the previous slide.

Consider the set-cover instance $U = \{1, 2, ..., 8\}$ and $\mathcal{S} = \{S_1, S_2, ..., S_5\}$ where $S_1 = \{1, 2, 3, 4\}$, $S_2 = \{2, 5, 7\}$, $S_3 = \{6, 7\}$, $S_4 = \{1, 8\}$, and $S_5 = \{1, 2, 3, 8\}$.

We can describe the instance with

$$\boldsymbol{M} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where each row corresponds to a set, and each column corresponds to an integer in U. The goal is again to find an optimal row cover!

Hence, if we have a ρ-approximate algorithm for hitting set, we can achieve approximation ratio ρ for set cover as well.