Dynamic Programming 3: Dependency

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Recall: Principle of dynamic programming

> Resolve subproblems according to a certain **order**. Remember the output of every subproblem to avoid re-computation.

Sometimes, the order may not be immediately obvious. To figure out a working order, we need to look at the subproblems' **dependency**.

Define a **string** as a sequence of characters.

A string s with length ℓ can be stored in an array of length ℓ.

We use $s[i]$ to represent the i-th char of s, for $i \in [1, \ell]$.

Given i, j s.t. $1 \leq i \leq j \leq \ell$, we use $s[i : j]$ to represent the sequence $s[i]s[i+1]...s[j]$, which is called a **substring** of s.

Example: If $s = \texttt{ABCD}$, then $s[2] = \texttt{B}$ and $s[2:4] = \texttt{BCD}$.

Problem

x = a string of length n

y = a string of length m

Consider function $f(i,j)$ defined for any $i \in [0,n]$ and $j \in [0,m]$:

$$f(i,j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ f(i-1,j-1)+1 & \text{if } i,j > 0 \text{ and } x[i] = y[j] \\ \max\{f(i,j-1), f(i-1,j)\} & \text{if } i,j > 0 \text{ and } x[i] \neq y[j] \end{cases}$$

Goal: Compute $f(n,m)$.

Example: Let $x = \texttt{ABC}$ and $y = \texttt{BDCA}$.
Then $f(2,1) = f(2,2) = f(2,3) = 1$ and $f(3,3) = f(3,4) = 2$.

A subproblem **depends** on another if the output of the latter is needed to solve the former.

$$f(i,j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ f(i-1,j-1)+1 & \text{if } i,j > 0 \text{ and } x[i] = y[j] \\ \max\{f(i,j-1), f(i-1,j)\} & \text{if } i,j > 0 \text{ and } x[i] \neq y[j] \end{cases}$$

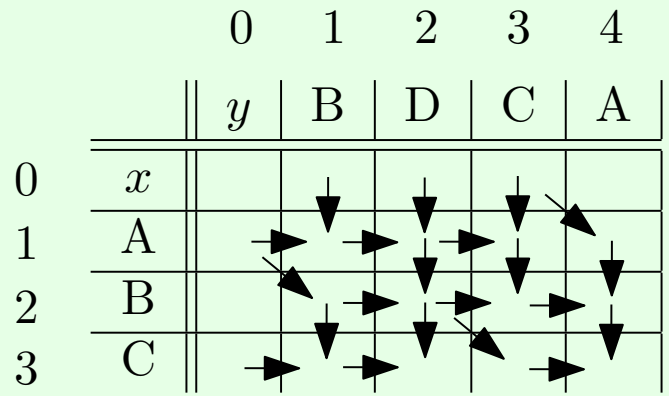
Subproblem $f(i,j)$ may depend on

- one subproblem $f(i-1,j-1)$ **or**
- two subproblems $f(i,j-1)$ and $f(i-1,j)$.

Which case it is depends on whether $x[i] = y[j]$.

Example: $x =$ ABC and $y =$ BDCA.

The dependency graph:



The cell at row $i \in [0, 3]$ and column $j \in [0, 4]$ represents subproblem $f(i, j)$.

For example, $f(3, 4)$ depends on $f(3, 3)$ and $f(2, 4)$, while $f(3, 3)$ depends only on $f(2, 2)$. Each arrow direction indicates the intended computation order, e.g., both $f(3, 3)$ and $f(2, 4)$ must be tackled before $f(3, 4)$.

To solve all subproblems with dynamic programming, we need to find a **topological order**.

This is an ordering of all subproblems satisfying the following condition: if subproblem A depends on subproblem B, then A must appear after B in the ordering.

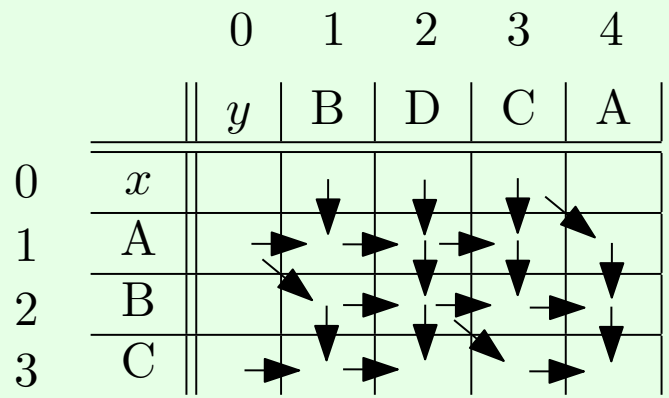
Any topological order can be deployed for dynamic programming.

For function

$$f(i,j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ f(i-1,j-1)+1 & \text{if } i,j > 0 \text{ and } x[i] = y[j] \\ \max\{f(i,j-1), f(i-1,j)\} & \text{if } i,j > 0 \text{ and } x[i] \neq y[j] \end{cases}$$

we can use either the row-major order or the column-major order.

Example: $x =$ ABC and $y =$ BDCA.



Row-major: $F(0,0)$, $F(0,1)$, ..., $F(0,4)$, $F(1,0)$, ..., $F(1,4)$, ...
Col-major: $F(0,0)$, $F(1,0)$, ..., $F(3,0)$, $F(0,1)$, ..., $F(3,1)$, ...

$$f(i,j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ f(i-1,j-1)+1 & \text{if } i,j > 0 \text{ and } x[i] = y[j] \\ \max\{f(i,j-1), f(i-1,j)\} & \text{if } i,j > 0 \text{ and } x[i] \neq y[j] \end{cases}$$

For any $f(i,j)$, we can compute it in $O(1)$ time, **given** the outputs of the subproblems it depends on.

We can therefore compute $f(n,m)$ in $O(nm)$ time
(the number of subproblems is $O(nm)$).