

CSCI3160: Regular Exercise Set 6

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Problem 1*. Let A be an array of n integers. Define a function $f(x)$ — where $x \geq 0$ is an integer — as follows:

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \max_{i=1}^x (A[i] + f(x-i)) & \text{otherwise} \end{cases}$$

Consider the following algorithm for calculating $f(x)$:

algorithm $f(x)$

1. **if** $x = 0$ **then return** 0
2. $max = -\infty$
3. **for** $i = 1$ **to** x
4. $v = A[i] + f(x - i)$
5. **if** $v > max$ **then** $max = v$
6. **return** max

Prove: the above algorithm takes $\Omega(2^n)$ time to calculate $f(n)$.

Solution. Let $g(x)$ denote the time of the algorithm in calculating $f(x)$. We know:

$$\begin{aligned} g(0) &\geq 1 \\ g(1) &\geq 1 \\ g(n) &\geq \sum_{i=0}^{n-1} g(i) \end{aligned}$$

We will show by induction that $g(n) \geq 2^{n-1}$ for $n \geq 1$. First, this is obviously correct when $n = 1$. Next, we will prove the claim on $n = k$ for any $k \geq 2$, assuming that it is correct for all $n \leq k - 1$.

$$\begin{aligned} g(n) &\geq \sum_{i=0}^{n-1} g(i) \\ &\geq 1 + \sum_{i=1}^{n-1} g(i) \\ &\geq 1 + \sum_{i=1}^{n-1} 2^{i-1} \\ &\geq 2^{n-1}. \end{aligned}$$

Problem 2. Let A be an array of n integers. Define function $f(a, b)$ — where $a \in [1, n]$ and $b \in [1, n]$ — as follows:

$$f(a, b) = \begin{cases} 0 & \text{if } a \geq b \\ (\sum_{c=a}^b A[c]) + \min_{c=a+1}^{b-1} \{f(a, c) + f(c, b)\} & \text{otherwise} \end{cases}$$

Design an algorithm to calculate $f(1, n)$ in $O(n^3)$ time.

Solution. We will launch n rounds. In the i -th round ($i \in [1, n]$), we calculate all the $f(a, b)$ satisfying $1 \leq a \leq b \leq n$ and $b = a + i - 1$. The strategy ensures that when $f(a, b)$ is computed, $f(a, c)$ and $f(c, b)$ are ready for all $c \in [a, b]$. Hence, the computation of $f(a, b)$ takes $O(n)$ time. The total running time is $O(n^3)$ because there are $O(n^2)$ values to compute.

Problem 3. In Lecture Notes 8, our algorithm for computing $f(n, m)$ has space complexity $O(nm)$, i.e., it uses $O(nm)$ memory cells. Reduce the space complexity to $O(n + m)$.

Solution. The lecture notes mentioned that we can list the subproblems in the “row-major” order. Specifically, row $i \in [0, n]$ contains all the subproblems $f(i, 0), f(i, 1), \dots, f(i, m)$; and we process the rows in ascending order of i . Storing all the rows consumes $O(nm)$ space. Noticing that only row $i - 1$ is needed to compute row $i \geq 1$. Therefore, at any moment, it suffices to store only two rows, which requires only $O(m)$ cells.

Remark: the space consumption is $O(n + m)$ (not $O(m)$) because you still need to store the input strings x and y .

Problem 4*. Let $G = (V, E)$ be a directed acyclic graph (DAG). For each vertex $u \in V$, let $\text{IN}(u)$ be the set of in-neighbors of u (recall that a vertex v is an in-neighbor of u if E has an edge from v to u). Define function $f : V \rightarrow \mathbb{N}$ as follows:

$$f(u) = \begin{cases} 0 & \text{if } \text{IN}(u) = \emptyset \\ 1 + \min_{v \in \text{IN}(u)} f(v) & \text{otherwise} \end{cases}$$

Design an algorithm to calculate $f(u)$ of every $u \in V$. Your algorithm should run in $O(|V| + |E|)$ time. You can assume that the vertices in V are represented as integers $1, 2, \dots, |V|$.

Solution. Compute a topological order of G in $O(|V| + |E|)$ time. Then, compute the $f(u)$ values of all vertices $u \in V$ according to the vertex ordering in the topological order.

Remark: Recall that a *topological order* of G is an ordering of the vertices in V where each vertex $u \in V$ is positioned after every vertex $v \in \text{IN}(u)$. A topological order can be obtained using depth first search in $O(|V| + |E|)$ time, which was discussed in CSCI2100. See Prof. Tao’s homepage (<http://www.cse.cuhk.edu.hk/~taoyf/>) for the course homepage of CSCI2100.

Problem 5.** Let $G = (V, E)$ be a directed acyclic graph (DAG). Design an algorithm to find the length of the longest path in G (recall that the length of a path is the number of edges in the path). Your algorithm should run in $O(|V| + |E|)$ time. You can assume that the vertices in V are represented as integers $1, 2, \dots, |V|$.

Solution. Define function $f : V \rightarrow \mathbb{N}$ as follows:

$$f(u) = \begin{cases} 0 & \text{if } \text{IN}(u) = \emptyset \\ 1 + \max_{v \in \text{IN}(u)} f(v) & \text{otherwise} \end{cases}$$

The length of the longest path equals $\max_{u \in V} f(u)$. Similar to Problem 4, we can compute $f(u)$ for all $u \in V$ in $O(|V| + |E|)$ time.