CSCI3160: Regular Exercise Set 10

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Problem 1. Let $G = (V, E)$ be a simple directed graph where each edge $(u, v) \in E$ carries a weight $w(u, v)$, which can be negative. Assume $V = \{1, 2, ..., n\}$ and G has no negative cycles. For any $u, v \in V$ and any integer $k \in [0, n]$, define $spdist(u, v| \le k)$ as the shortest length of any path from u to v where all the intermediate vertices are in $\{1, 2, ..., k\}$ (recall: an *intermediate vertex* is a vertex on the path other than the source and destination vertices). Prove:

$$spdist(u, v \mid\le k) = \min \begin{cases} spdist(u, k \mid\le k-1) \\ spdist(u, k \mid\le k-1) + spdist(k, v \mid\le k-1) \end{cases}$$

(Hint: First prove LHS $\le$ RHS and then prove LHS $\ge$ RHS.)

Solution. We prove only LHS $\ge$ RHS (the other direction is obvious and left to you). Consider a path π from u to v that uses intermediate vertices only from $\{1, 2, ..., k\}$ and has length $spdist(u, v \mid\le k)$. If k is not an intermediate vertex of π, then the length of π is at least $spdist(u, k \mid\le k-1)$, and we have LHS $\ge$ RHS. Next, we consider the scenario where k is an intermediate vertex of π.

Suppose that k appears only once on π. In this case, we can break π into: (i) π_1, which is the prefix of π from u to k, and (ii) π_2, which is the suffix of π from k to v. Both π_1 and π_2 use intermediate vertices only from $\{1, 2, ..., k-1\}$; hence, their lengths are at least $spdist(u, k \mid\le k-1)$ and $spdist(k, v \mid\le k-1)$, respectively. This implies that LHS $\ge$ RHS.

If k appears more than once on π, it means that π contains at least one cycle from k to itself. We can remove all cycles from π, which can only decrease the length of π (no negative cycles). With all cycles removed, we are left with a path from u to v where k appears only once. Then, our earlier argument applies.

Problem 2. Let $G = (V, E)$ be a simple directed graph where each edge $(u, v) \in E$ carries a weight $w(u, v)$, which can be negative. Assume $V = \{1, 2, ..., n\}$. This time, G may or may not have a negative cycle. Given any $u, v \in V$ (where u and v can be identical), we define a *simple path* from u to v as a path π satisfying:

- π starts from u and ends at v;
- u is not an intermediate vertex of π;
- v is not an intermediate vertex of π;
- no intermediate vertex appears twice on π.

For any $u, v \in V$ and any integer $k \in [0, n]$, define $spdist(u, v| \le k)$ as the shortest length of any *simple* path from u to v where all the intermediate vertices are in $\{1, 2, ..., k\}$. Prove:

$$spdist(u, v \mid\le k) = \min \begin{cases} spdist(u, k \mid\le k-1) \\ spdist(u, k \mid\le k-1) + spdist(k, v \mid\le k-1) \end{cases}$$

(Hint: First prove LHS $\le$ RHS and then prove LHS $\ge$ RHS.)

Solution. Our proof for Problem 1 applies verbatim after removing the proof's last paragraph.

Problem 3. Let $G = (V, E)$ be a simple directed graph where each edge $(u, v) \in E$ carries a weight $w(u, v)$, which can be negative. Assume $V = \{1, 2, ..., n\}$. G may or may not have a negative cycle. For any $u, v \in V$ and an integer $k \in [0, n]$, define $\mathit{spdist}(u, v| \le k)$ as the shortest length of any <u>*simple*</u> path (defined as in Problem 3) from u to v where all the intermediate vertices are in $\{1, 2, ..., k\}$. Prove: G has a negative cycle if and only if $\mathit{spdist}(u, u| \le n) < 0$ for some $u \in V$.

Remark. This implies that The Floyd-Warshall algorithm can detect the presence of negative cycles in $O(|V|^3)$ time.

Solution. The $\Rightarrow$ direction. Consider a negative cycle C. Let u be the *largest* vertex on C (recall that each vertex is an integer in $\{1, 2, ..., n\}$), and v be any other vertex on C. Define π_1 as the path from u to v on C, and π_2 as the path from v to u on C. Note that π_1 and π_2 use intermediate vertices only from $\{1, 2, ..., n-1\}$. Therefore, the length of π_1 is at least $\mathit{spdist}(u, v| \le n-1)$ and the length of π_2 is at least $\mathit{spdist}(v, u| \le n-1)$. It thus follows from the result of Problem 2 that

$$\begin{aligned} \mathit{spdist}(u, u| \le n) &\le \mathit{spdist}(u, v| \le n-1) + \mathit{spdist}(v, u| \le n-1) \\ &\le \text{length of } C \\ &< 0. \end{aligned}$$

The $\Leftarrow$ direction. Assume that G has no negative cycles. We claim that $\mathit{spdist}(u, u| \le n)$ must be 0 for every $u \in V$, which thus gives a contradiction. First, $\mathit{spdist}(u, u| \le n)$ cannot be positive because u can obviously "reach" itself with length 0. Second, if $\mathit{spdist}(u, u| \le n)$ is negative, then it indicates the existence of a negative cycle.

Problem 4. Let $G = (V, E)$ be a simple directed graph where every edge (u, v) carries a weight $w(u, v)$, which can be negative. G has no negative cycles. Recall that Johnson's algorithm adds a vertex v_{dummy} to G and computes the shortest path distance $\mathit{spdist}(v_{dummy}, v)$ from v_{dummy} to every vertex. Then, the weight of each edge (u, v) is modified to:

$$w'(u, v) = w(u, v) + \mathit{spdist}(v_{dummy}, u) - \mathit{spdist}(v_{dummy}, v).$$

Prove: $w'(u, v) \ge 0$.

Solution. The claim $w'(u, v) \ge 0$ is equivalent to $w(u, v) + \mathit{spdist}(v_{dummy}, u) \ge \mathit{spdist}(v_{dummy}, v)$. The latter inequality holds because $w(u, v) + \mathit{spdist}(v_{dummy}, u)$ gives the length of only one path from v_{dummy} to v, and therefore, is at least the shortest distance $\mathit{spdist}(v_{dummy}, v)$ from v_{dummy} to v.

Problem 5. Let $G = (V, E)$ be a simple directed graph where every edge (u, v) carries a *non-negative* weight $w(u, v)$. Apply Johnson's algorithm to compute a new weight $w'(u, v)$ for each edge $(u, v) \in E$. Prove: $w'(u, v) = w(u, v)$.

Solution. Follows immediately from the fact that $\mathit{spdist}(v_{dummy}, v) = 0$ for every $v \in V$.