Exercises

Problem 1. Let P be a set of points in $\mathbb{R}^2$. Denote by n the size of P and by k the number of points on the convex hull boundary of P. In an earlier exercise, we proved that any triangulation of P must contain $2n-2-k$ triangles. Use this result to prove: any triangulation must contain $3n-3-k$ edges.

(Hint: Euler formula.)

Problem 2. Let P be a set of 2D points in general position. Prove: by connecting each pair of Voronoi neighbors with a line segment, we obtain a triangulation of P.

Remark: this proves the dual relationship between the Voronoi diagram and Delaunay triangulation.

(Hint: those segments define an SLPG. Prove every Voronoi vertex corresponds to a distinct internal face in the SLPG.)

Problem 3. Let P be a set of 2D points in general position. Suppose that we have obtained a triangulation of P such that the circle of every triangle contains no point of P in the interior. Then, this triangulation must be the Delaunay triangulation.

(Hint: use the result of Problem 1.)

Problem 4. Consider using the algorithm discussed in the class to compute the Delaunay triangulation of a set P of n 2D points. Recall that the algorithm incrementally inserts each point in P and that each insertion will introduce a number of diagonals. Prove: the number of diagonals introduced at every insertion is $O(1)$ in expectation (provided that the insertion order is a random permutation of P).

(Hint: the Delaunay triangulation has a linear complexity. Use this property in backward analysis.)

Problem 5 (all nearest neighbors). Let P be a set of n points in $\mathbb{R}^2$. The nearest neighbor of a point $p \in P$ is the point in $P \setminus \{p\}$ with the smallest Euclidean distance to p. Give an algorithm to find the nearest neighbors of all points in P. Your algorithm needs to finish in $O(n \log n)$ expected time.

Problem 6* (Exercise 9.11 from the textbook). Let P be a set of n points in $\mathbb{R}^2$. A Euclidean spanning tree of P is a tree where every vertex is a point in P and every edge is a line segment connecting two points of P. The tree's *weight* equals the total (Euclidean) length of all the segments. The *Euclidean minimum spanning tree* (EMST) is a Euclidean spanning tree of the smallest weight.

- Prove: the Delaunay triangulation of P contains an EMST for P.
- Give an algorithm to find an EMST in $O(n \log n)$ expected time.

(Hint: think about how Kruskal's algorithm runs on the complete graph.)