SQ 33.

Aim: To understand the original Stern-Gerlach Experiment based on spin angular momentum

(a) Calculate the z-component of $\vec{\mu}_s$.

Spin magnetic dipole moment $\vec{\mu}_s = -\frac{e}{m}\vec{S}$.

If you consider the z-component, then

$\hat{\mu}_{sz} = -\frac{e}{m}\hat{S}_z$

Since $\hat{S}_z$ has the eigenvectors $|\uparrow\rangle$ and $|\downarrow\rangle$ correspoinding to eigenvalues of $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$, $\hat{\mu}_{sz}$ also has the same eigenvalues and eigenvectors:

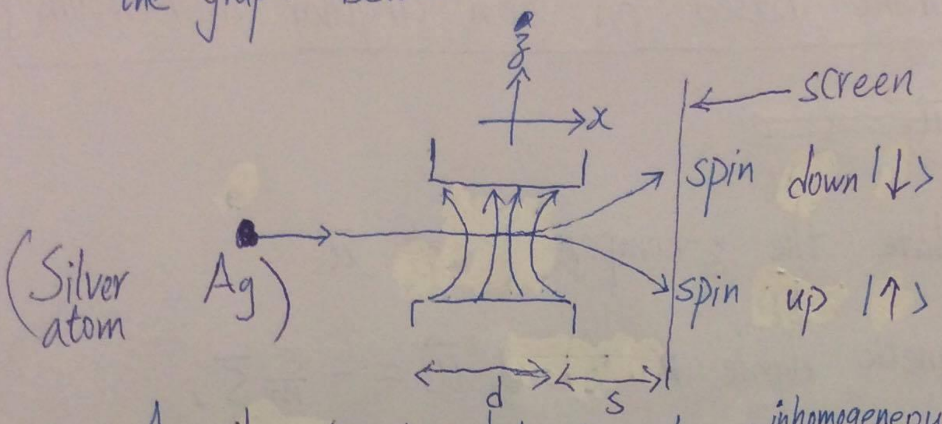
$$\therefore \begin{cases} \hat{\mu}_{sz}|\uparrow\rangle = -\frac{e}{m}\hat{S}_z|\uparrow\rangle = -\frac{e\hbar}{2m}|\uparrow\rangle \\ \hat{\mu}_{sz}|\downarrow\rangle = -\frac{e}{m}\hat{S}_z|\downarrow\rangle = \frac{e\hbar}{2m}|\downarrow\rangle \end{cases}$$

Define $\mu_B = \frac{e\hbar}{2m}$. We call μ_B the Bohr magneton.

$$\therefore \begin{cases} \hat{\mu}_{sz}|\uparrow\rangle = -\mu_B|\uparrow\rangle \\ \hat{\mu}_{sz}|\downarrow\rangle = \mu_B|\downarrow\rangle \end{cases}$$

$\therefore$ The z-component of $\vec{\mu}_s$ can take on $\pm\mu_B$.

(b) In this question, we have a situation indicated by the graph below.



A silver atom is shot towards the inhomogeneous magnetic field, nearly perpendicular to the motion. Depending on the direction of the spin of outermost electron, the silver atom is bent either upwards or downwards by the "magnetic force" acting on the spin.

Information: Kinetic energy of the Ag atom $= 3\times10^{-20}\,J$,

$\frac{\partial B}{\partial z} = 2.3\times10^{3}\ T\,m^{-1}$,

The seperation of magnets and the screen $(s)=0.25\,m$

The distance travelled in the magnetic field $(d)=0.03\,m$

For spin up Ag,

$$\mu_{sz} = -\mu_B \quad (\hat{\mu}_{sz}|\uparrow\rangle = -\mu_B|\uparrow\rangle)$$

$$F_\uparrow = -\mu_B \frac{\partial B}{\partial z}$$

For spin down Ag,

$$\mu_{sz} = \mu_B \quad (\hat{\mu}_{sz}|\downarrow\rangle = \mu_B|\uparrow\rangle)$$

$$F_\downarrow = \mu_B \frac{\partial B}{\partial z}$$

Notice that there is no concept of force in quantum mechanics. Classical force is used to simpify the problem.

For spin down Ag,

it is bent upwards.

Before entering the magnetic field, it has velocity $V_x = \sqrt{\frac{2(K.E)}{m}}$ $\left(\frac{1}{2}mV_x^2 = K.E.\right)$

$$= \sqrt{\frac{2(3\times10^{-20})}{(1.79\times10^{-25})}}$$

$$= 578.96\ ms^{-1}$$

time travelled in magnetic field (Δt_1)

$$= \frac{d}{V_x} = \frac{0.03}{578.96} = 5.182\times10^{-5}\ s$$

Momentum gained by the spin down Ag atom (in y-direction)

$$= F_{\downarrow}\Delta t = \mu_B \frac{\partial B}{\partial z}\Delta t = (9.274\times10^{-24})(2.3\times10^{3})(5.182\times10^{-5})$$

$$= 1.105\times10^{-24}\ kg\ ms^{-1}$$

$$= mV_y$$

$$V_y = \frac{1.105\times10^{-24}}{1.79\times10^{-25}} = 6.173\ ms^{-1}.$$

The vertical distance it travelled

$$= (6.173)\frac{(0.25)}{(578.96)} = 2.67\times10^{-3}\ m$$

$$= 2.67\ mm\ (\text{upward}).$$

Similarly, the spin up Ag will travel 2.67 mm (downward)

The separation of two beams 0.25m beyond the magnet.

$= 2.67 \times 2 = 5.34$ mm.

SQ 34)

(a) $\hat{S_y} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$S_y \vec{v} = \lambda \vec{v}$, λ is eigenvalue, $\vec{v}$ is eigenstate

$$\Rightarrow \begin{vmatrix} -\lambda & -i\frac{\hbar}{2} \\ i\frac{\hbar}{2} & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - \frac{\hbar^2}{4} = 0$$

$$\lambda = +\frac{\hbar}{2} \text{ or } -\frac{\hbar}{2}$$

For $\lambda = +\frac{\hbar}{2}$,

$$\begin{pmatrix} -\frac{\hbar}{2} & -i\frac{\hbar}{2} \\ i\frac{\hbar}{2} & -\frac{\hbar}{2} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{cases} -v_1 - iv_2 = 0 \\ iv_1 - v_2 = 0 \end{cases}$$

$\therefore v_1 = 1$ and $v_2 = i$

For $\lambda = -\frac{\hbar}{2}$,

$$\begin{pmatrix} \frac{\hbar}{2} & -i\frac{\hbar}{2} \\ i\frac{\hbar}{2} & \frac{\hbar}{2} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{cases} v_1 - iv_2 = 0 \\ iv_1 + v_2 = 0 \end{cases}$$

$\therefore v_1 = 1$ and $v_2 = -i$

Normalization factor: $\frac{1}{C} = \sqrt{(1+i)(1-i)}$

$$C = \frac{1}{\sqrt{2}}$$

$\therefore$ For $\lambda = +\frac{\hbar}{2}$, $|\uparrow\rangle_y = \alpha_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$; For $\lambda = -\frac{\hbar}{2}$, $|\downarrow\rangle_y = \beta_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$

(b) $\alpha_y^* \beta_y = \frac{1}{2}(1 \; -i)\begin{pmatrix} 1 \\ -i \end{pmatrix}$

$$= \frac{1}{2}(1-1)$$

$$= 0$$

$\therefore$ By definition, α_y and β_y are orthogonal to each other.

(c) For $\alpha_z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ with $S_z = \frac{\hbar}{2}$,

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = C_1 \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} + C_2 \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$= \begin{pmatrix} C_1' + C_2' \\ iC_1' - iC_2' \end{pmatrix}$$ where $C_1' = \frac{C_1}{\sqrt{2}}$ and $C_2' = \frac{C_2}{\sqrt{2}}$

$$\Rightarrow \begin{cases} C_1' + C_2' = 1 \\ C_1' - C_2' = 0 \end{cases}$$

$\therefore C_1' = C_2' = \frac{1}{2} \Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\alpha_y + \frac{1}{\sqrt{2}}\beta_y$

As $c_1' = c_2'$, the probability of getting $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$ are equal, $\langle \hat{S_y} \rangle = 0$

Mathematically,

$$\langle \hat{S_y} \rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ i \end{pmatrix}$$

$$= 0$$

$$\langle \hat{S_y}^2 \rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{\hbar^2}{4} \begin{pmatrix} 0 & -i \end{pmatrix} \begin{pmatrix} 0 \\ i \end{pmatrix}$$

$$= \frac{\hbar^2}{4}$$

$$(\Delta S_y) = \sqrt{\langle \hat{S_y}^2 \rangle - \langle \hat{S_y} \rangle^2}$$

$$= \frac{\hbar}{2}$$

For $\beta z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with $S_z = -\frac{\hbar}{2}$,

$$\langle \hat{S_y} \rangle = \begin{pmatrix} 0 & 1 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} -i \\ 0 \end{pmatrix}$$

$$= 0$$

$$\langle \hat{S_y}^2 \rangle = \begin{pmatrix} 0 & 1 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{\hbar^2}{4} \begin{pmatrix} i & 0 \end{pmatrix} \begin{pmatrix} -i \\ 0 \end{pmatrix}$$

$$= \frac{\hbar^2}{4}$$

$$\langle \Delta S_y \rangle = \sqrt{\langle \hat{S_y}^2 \rangle - \langle \hat{S_y} \rangle^2}$$

$$= \frac{\hbar}{2}$$

$\therefore$ Both have the same $\langle \hat{S_y}^2 \rangle$ and ΔS_y

(c) For a general state $\begin{pmatrix} c \\ d \end{pmatrix}$,

$$\langle \hat{S_y} \rangle = (c \ d) \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix}$$

$$= \frac{\hbar}{2} (c \ d) \begin{pmatrix} -id \\ ic \end{pmatrix}$$

$$= \frac{\hbar}{2} (-icd + icd)$$

$$= 0$$

$$\begin{pmatrix} c \\ d \end{pmatrix} = C_1 \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} + C_2 \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$= \begin{pmatrix} C_1' + C_2' \\ iC_1' - iC_2' \end{pmatrix} \quad \text{where } C_1' = \frac{C_1}{\sqrt{2}} \text{ and } C_2' = \frac{C_2}{\sqrt{2}}$$

$$\Rightarrow \begin{cases} c = C_1' + C_2' \\ d = iC_1' - iC_2' \end{cases} \Rightarrow \begin{cases} c = C_1' + C_2' \\ id = -C_1' + C_2' \end{cases}$$

$$c + id = 2C_2'$$

$$C_2' = \frac{c+id}{2}$$

$$c - id = 2C_1'$$

$$C_1' = \frac{c-id}{2}$$

$$\therefore \begin{pmatrix} c \\ d \end{pmatrix} = \underbrace{\frac{c-id}{\sqrt{2}}}_{\sqrt{P} \text{ of getting } +\frac{\hbar}{2}} \alpha_y + \underbrace{\frac{c+id}{\sqrt{2}}}_{\sqrt{P} \text{ of getting } -\frac{\hbar}{2}} \beta_y$$

$$\left| \frac{c-id}{\sqrt{2}} \right| = \frac{1}{\sqrt{2}} [(c-id)(c+id)]^{-\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2}} (c^2 + d^2)^{\frac{1}{2}} = \left| \frac{c+id}{\sqrt{2}} \right|$$

$\therefore$ Sometimes we get $+\frac{\hbar}{2}$ and sometimes we get $-\frac{\hbar}{2}$, but their probability are equal! $\langle \hat{S_y} \rangle = 0$

(d) $\alpha_y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$ with $S_y = \frac{\hbar}{2}$,

$$\langle \hat{S_z} \rangle = \frac{1}{2}\begin{pmatrix}1 & -i\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1\\i\end{pmatrix}$$

$$= \frac{\hbar}{4}\begin{pmatrix}1 & -i\end{pmatrix}\begin{pmatrix}1\\-i\end{pmatrix}$$

$$= \frac{\hbar}{4}(1-1)$$

$$= 0$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} = C_1\begin{pmatrix}1\\0\end{pmatrix} + C_2\begin{pmatrix}0\\1\end{pmatrix}$$

$$= \begin{pmatrix}C_1\\C_2\end{pmatrix}$$

$$\Rightarrow \begin{cases} C_1 = \frac{1}{\sqrt{2}} \\ C_2 = \frac{i}{\sqrt{2}} \end{cases}$$

$$|C_1|^2 = |C_2|^2 = \frac{1}{2}$$

$\therefore$ Same as before, half of the chance getting $+\frac{\hbar}{2}$ and half of the chance getting $-\frac{\hbar}{2}$! $\langle \hat{S_z} \rangle = 0$.