DEPARTMENT OF PHYSICS, CHINESE UNIVERSITY OF HONG KONG
PHYS3021 QUANTUM MECHANICS I

SAMPLE QUESTIONS FOR WEEK 7 EXERCISE CLASSES (16-20 October 2017)

Read me: TA will discuss the **SAMPLE QUESTIONS** (SQs) and answer your questions in exercise classes every week. The SQs are designed to review what you have learnt, tell a physics story, enrich our discussions, and help you work out the upcoming Problem Set. **Your time table should allow you to attend one exercise class session. The Exercise Classes are an integrated part of the course.** You are encouraged to work out (or think about) the SQs before attending exercise class and ask the TA questions. Over the semester, you are welcome to seek help from TAs/me.

SQ17 - Fourier Transform is expanding a function in terms of momentum eigenfunctions
SQ18 - Be very careful in measurements and calculations - See if this can confuse you

SQ17 *Fourier Transform is the special case of expanding a function in terms of momentum eigenfunctions*

In class, we discussed that we could use the eigenfunctions of any quantity represented by $\hat{A}$ to expand a function. Here, we look at the special case where $\hat{A} = \hat{p}$, the momentum operator.

Consider the momentum operator $\hat{p} = \frac{\hbar}{i}\frac{d}{dx}$. **Show that** it has eigenfunctions of the form

$$\psi_k(x) = \frac{1}{\sqrt{2\pi}}\, e^{ikx} \tag{1}$$

with eigenvalues $p = \hbar k$. For later purposes, the eigenfunctions are labelled by k instead of by p, but they are only off by a constant. The prefactor $\frac{1}{\sqrt{2\pi}}$ is inserted for convenience (stay tuned).

Note that the eigenvalues k are **not discrete**, but take on **continuous values**, i.e., any k is allowed. The reason is simple: The momentum eigenvalue problem does not imply a boundary (or it is free particle on x-axis). Thus, there is no boundary condition to select discrete eigenvalues. A by-product is that it is not a bound state. Being **not a bound state**, the eigenfunctions **cannot be normalized** in the usual way. There are several ways out, and each way has its merits. One way is to invoke something called the **Dirac δ-function**. Pictorially, $\delta(x-a)$ is (i) zero everywhere for $x \neq a$, (ii) an extremely sharp spike at $x = a$, (iii) symmetric about $x = a$, and (iv) having an area under the whole function and thus under the sharp spike (which is an integral) being one. The most useful relation (or definition) of the δ-function is

$$\int_{-\infty}^{+\infty} f(x)\,\delta(x-a)dx = f(a) \tag{2}$$

for any function $f(x)$. The RHS is the function's value at $x = a$, i.e., at the point of the spike of the δ-function. If $f(x) = 1$ (a boring function), it is property (iv) stated above. One can immediately think of different ways to represent (called representations) the δ-function. For example, a rectangle of width W and height $1/W$ centered at $x = a$ with $W \to 0$ will do the job. Here, you will see another representation. This ends an introduction to the Dirac δ-function.

Recall the **Fourier Transform** of a function $f(x)$ is

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \phi(k)e^{ikx}dk \tag{3}$$

Consider the case $f(x) = e^{ik'x}$ (which is a function of x, right), using Eq.(2) to **show** that

$$\phi(k) = \sqrt{2\pi}\,\delta(k-k') \tag{4}$$

But $\phi(k)$ can also be found by the Inverse Fourier Transform formula. **Show that**

$$\int_{-\infty}^{+\infty} e^{ikx}e^{-ik'x}dx = 2\pi\delta(k'-k) \tag{5}$$

Hence, **show** that a modified way of expressing the "orthonormal" property of the momentum eigenfunctions $\psi_k(x)$ in Eq. (1) is

$$\int_{-\infty}^{+\infty} \psi^*_{k'}(x)\psi_k(x)dx = \delta(k'-k) = \delta(k-k') \tag{6}$$

Note that for $k \neq k'$ (thus different momentum eigenfunctions), the RHS is zero, thus orthogonal. For $k = k'$, the RHS is what a δ-function will behave. This reflects the fact that $\psi_k(x)$ is not a bound state and cannot be normalized to one. The Dirac δ-function comes into rescue and "normalize" them differently.

Finally, **illustrate** that the Fourier Transform and Inverse Fourier Transform are simply the formulas of expanding any function $\Phi(x)$ in terms of the momentum eigenfunctions $\psi_k(x)$.

[Remark: You see here that the momentum operator is very powerful. Its eigenfunctions can be used to expand any function, as Fourier told us 200 years ago!]

SQ18 *Be very careful in measurements and calculations - See if this can confuse you*

We stressed in class the physical meaning of the expectation value $\langle A \rangle$ and the uncertainty (ΔA) is related to measurements on identical copies of a system and each copy is measured only once. This SQ also follows-up Problems 2.4 and 2.5.

An experimentalist prepared a huge number of identical copies of a state of the form

$$\Psi(x) = \sqrt{\frac{1}{3}}\,\psi_0(x) + \sqrt{\frac{2}{3}}\,\psi_1(x) \tag{7}$$

where $\psi_0(x)$ is the normalized ground state wavefunction of a harmonic oscillator (as used in Problem 2.5) and $\psi_1(x)$ is the first excited state wavefunction.

Case A: A student Alice separates the copies into three equal parts. For 1/3 of the copies, measurements on the energy E are made (system thrown away after measured once). For another 1/3 of the copies, she measures the position x (and throws copy away after a measurement). For the remaining 1/3, measurements on momentum p are made.

(a) **What** are the mean energy $\langle E \rangle$ and the uncertainty (ΔE)? **Explain** why we need to consider these quantities in light of the distribution of the energy measurement outcomes.

(b) **What** will Alice get for $\Delta x \cdot \Delta p$?

Case B: A student Bob first measures the energy of each copy. He keeps the copies after measurements. He then sorts the copies into two groups according to the energy outcomes. For Group 0 of energy E_0, he measures the position on half of the group (one measurement per copy) and the momentum on the other half of the group (one measurement per copy). He does the same for the Group 1 of energy E_1.

(c) **Find** $\Delta x \cdot \Delta p$ for Group 0.

(d) **Find** $\Delta x \cdot \Delta p$ for Group 1.

(e) But Bob's supervisor doesn't think that there should be two values of $\Delta x \cdot \Delta p$. So Bob thought that he should take into account of the sample sizes of Group 0 and Group 1 and do some weighting/averages. **Try** one or two reasonable weighting schemes and **compare** results with part (b).

(f) **Point out** why $\Delta x \cdot \Delta p$'s in (b), (c), (d) are different and what they really refer to?