DEPARTMENT OF PHYSICS, CHINESE UNIVERSITY OF HONG KONG
PHYS3021 QUANTUM MECHANICS I

SAMPLE QUESTIONS FOR WEEK 3 EXERCISE CLASSES (18-22 September 2017)

Read me: TA will discuss the **SAMPLE QUESTIONS** (SQs) and answer your questions in exercise classes every week. The SQs are designed to review what you have learnt, tell a physics story, enrich our discussions, and help you work out the upcoming Problem Set. **Your time table should allow you to attend one exercise class session. The Exercise Classes are an integrated part of the course.** You are encouraged to work out (or think about) the SQs before attending exercise class and ask the TA questions. Over the semester, you are welcome to seek help from TAs/me.

SQ8 - Normalizing a wavefunction and applying probabilistic interpretation
SQ9 - Fourier transform decomposes a wavefunction into states of definite momenta
SQ10 - Decomposing a wavefunction can be carried out using different sets of functions

SQ8 *Normalizing a wavefunction and Applying the probabilistic interpretation*

Let's say we have a wave form that $\psi(x) = 0$ for $x < 0$ and $x > a$. Between $0 \leq x \leq a$. The form is like a tent (or looks like Λ). It goes linearly up from $\psi(0) = 0$ to $x = a/2$ and then goes linearly down to $\psi(a) = 0$.

[Remark: Formally, we need not know the physical context for which this form is of practical use. You saw the QM 1D particle-in-a-box problem in PHYS1122 in which a particle is confined to live in $0 \leq x \leq a$ by two infinite potential walls. This wavefunction satisfies the requirement that the particle cannot be found outside the box and that it is continuous, although there is something unusual at $x = a/2$. In classical physics, it can describe a string of length a and fixed at the two ends being held in the shape of a tent.]

(a) In QM, a wavefunction describing a particle can be normalized. **Determine** $\psi(\frac{a}{2})$ by requiring that the wave function is normalized. Hence, **sketch** $\psi(x)$ for the whole range of x (not only for $0 \leq x \leq a$). It should be emphasized that the two zero tails are also parts of the whole function $\psi(x)$.

(b) Born (Bohr and Heisenberg) interpreted $|\psi(x)|^2 dx$ as the probability of finding the particle to be in the interval x to $x + dx$. **Sketch** $|\psi(x)|^2$ as a function of x. Based on Born's interpretation, **find** the mean position $\langle x \rangle$. **Discuss the physical meaning** of the mean position in an experimental viewpoint. [Hint: Recall that we need to **repeat the experiment** of sending in one electron at a time into the two-slit experiment **many times** in order to see the interference pattern.]

(c) **Find** the probability of finding the particle in the range $2a/5 < x < 3a/5$.

(d) **Introduce** the variance $\sigma_x^2 = \langle (x - \langle x \rangle)^2 \rangle \equiv (\Delta x)^2$ of position to students and **evaluate** it. [Students: Now you see that there is a concrete way to evaluate the uncertainty in position.]

SQ9 *Taking $\psi(x)$ as a wave packet and analyzing its Fourier components*

Looking at $\psi(x)$ in SQ8, it is obviously a wave packet, i.e., it describes a particle with a range to be found, but certainly not spreading out all over the place (all over x). As discussed in class, we know that the wider the wave, the fewer are the Fourier components (labelled by k talking about mathematics and related to the momentum p talking about physics), and vice versa. Here, TA will analyze $\psi(x)$ in Fourier's way.

There are many ways to write the Fourier transform and Inverse Fourier transform. Let's use the symmetric way, i.e.,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k)\, e^{ikx}\, dk \tag{1}$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx \tag{2}$$

Eq. (1) says that any (well-behaved) $f(x)$ can be formed by adding up plane waves of different wavevectors k provided that the weighting $F(k)$ are chosen appropriately. Eq. (2) tells us how the choose $F(k)$ appropriately.

TA: **Find** $F(k)$ for the given form of $\psi(x)$ in SQ8. Just like the QM wavefunction, $F(k)$ is in general complex. Hence, **Sketch** $|F(k)|^2$ and **re-iterate** the meaning of $F(k)$ being the weighting of the component e^{ikx} in $\psi(x)$. [Remark: Had the range of x been chosen as $-a/2 \leq a \leq a/2$ and thus the peak is at $x = 0$, $F(k)$ will be real, but $|F(k)|$ or $|F(k)|^2$ is just the same. Students are encouraged to try it out.]

A Twist - Getting something from nothing. De Broglie said that $\lambda = h/p$, thus a particle of **definite momentum** p has a definite wavelength and thus a definite $k = 2\pi/\lambda = p/\hbar$. Hence, $\psi_p(x) \sim e^{ikx} \sim e^{ipx/\hbar}$ corresponds to the wavefunction of a state of definite momentum p. From Eq. (1), if the integration over k is changed to an integration over p, a factor of $1/\hbar$ will come out because $p = \hbar k$. To make the two expressions symmetric again, we will put a factor of $1/\sqrt{\hbar}$ in Eq. (1) and Eq. (2). Eq. (1) and Eq. (2) then become

$$\psi(x) = \int_{-\infty}^{\infty} F(p) \qquad \underbrace{\frac{1}{\sqrt{2\pi\hbar}}\, e^{ipx/\hbar}}_{\text{state of definite momentum } p} \qquad dp \tag{3}$$

and

$$F(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x)\, e^{-ipx/\hbar}\, dx \tag{4}$$

Write down $F(p)$ and $|F(p)|^2$ using the previous result. **Hence** discuss the behavior of $|\psi(x)|^2$ and $|F(p)|^2$ for the cases of a bigger a and a smaller a.

[Remarks: Eqs. (3) and (4) are important. They show the way to analyze any wavefunction $\psi(x)$ in terms of **states of definite momentum** (to be called **momentum eigenstates**). We discussed the measurement interpretation of $|\psi(x)|^2$ in SQ8. In fact, there is also a measurement interpretation of $|F(p)|^2$, but (TA: don't do that) we will hold on to that until you discover the interpretation by yourself later.]

SQ10 *Expanding $\psi(x)$ by another set of functions*

Here is another twist of SQ8 and SQ9. Consider $\psi(x)$ in SQ8 again. Let's consider the set of (infinitely many) functions defined by $\phi_n(x) = 0$ in $x < 0$ and $x > a$ for all n, and

$$\phi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \quad \text{for } 0 \leq x \leq a \text{ and } n = 1, 2, \ldots \tag{5}$$

We want to express $\psi(x)$ as a linear combination of $\phi_n(x)$, i.e.,

$$\psi(x) = \sum_{n=1}^{\infty} a_n \phi_n(x) \tag{6}$$

TA: **Show** the way to determine the coefficients a_n.

[Remarks: Interestingly, the same $\psi(x)$ can be expanded in terms of different sets of functions. In SQ9, we used the functions corresponding to definite momenta. Here, we used the set of Eq. (5), which (as you may know) are the states of definite energies for the particle-in-a-box problem. There is also a measurement interpretation for $|a_n|^2$ (but hold on to it). The mathematical skill for determining a_n in this SQ is important in applying the time-dependent Schrödinger equation to solve **initial value problems**, as will be discussed in Chapter III.]