DEPARTMENT OF PHYSICS, CHINESE UNIVERSITY OF HONG KONG
PHYS3021 QUANTUM MECHANICS I

SAMPLE QUESTIONS FOR WEEK 10 EXERCISE CLASSES (6 - 10 November 2017)

Read me: TA will discuss the **SAMPLE QUESTIONS** (SQs) and answer your questions in exercise classes every week. The SQs are designed to review what you have learnt, tell a physics story, enrich our discussions, and help you work out the upcoming Problem Set. **Your time table should allow you to attend one exercise class session. The Exercise Classes are an integrated part of the course.** You are encouraged to work out (or think about) the SQs before attending exercise class and ask the TA questions. Over the semester, you are welcome to seek help from TAs/me.

SQ24 - Transforming 2D Laplacian into plane polar coordinates
SQ25 - A state of definite L_z has uncertain L_y and $\langle L_y \rangle = 0$
SQ26 - 2D isotropic oscillator in plane polar coordinates (example)

SQ24 *Writing 2D Laplacian in plane polar coordinates*

This is to prepare for a question in Problem Set 4. It is an exercise on partial differentiation. In 2D problems, we can either use the Cartesian coordinates on the plane polar coordinates. Particularly, when the potential energy function is "circularly symmetric", then it is more convenient to use plane polar coordinates.

Start with the relationship between (x, y) in Cartesian coordinates and (r, ϕ) in plane polar coordinates, **transform** the operator

$$\nabla^2_{2D} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{1}$$

into plane polar coordinates.

SQ25 *Spherical Harmonics are very useful – A state of definite L_z has uncertain L_y and $\langle L_y \rangle = 0$*

[You will do a similar problem in Problem Set 4.]

In discussing orbital angular momentum, we stressed that (after inspecting the commutators) we can at best know the magnitude squared and one component. For convenience and without loss of generality, we choose the component to be the z-component. This SQ supplements the discussion.

(a) Following the same procedure in SQ24, one can transform Cartesian coordinates to spherical coordinates in 3D. In particular. **Demonstrate** how $\hat{L}_y$ can be written in spherical coordinates. [$\hat{L}_z$ was done in class notes.]

(b) The nice feature of $Y_{\ell m_\ell}(\theta, \phi)$ is that they are simultaneous eigenstates of $\hat{L}^2$ and $\hat{L}_z$. Take $Y_{11}(\theta, \phi)$ for example (look up the functional form), it has definite values of $L^2 = 2\hbar^2$ (or $L = \sqrt{2}\hbar$) and $L_z = \hbar$. Since $[\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x \neq 0$, a state of definite L_z should have uncertain L_y. The first thing to notice is that an eigenstate of $\hat{L}_z$ is NOT an eigenvalue of $\hat{L}_y$.

TA: To demonstrate this point, **operator** $\hat{L}_y$ on $Y_{11}(\theta, \phi)$ and **show** that Y_{11} is NOT an eigenstate of $\hat{L}_y$.

(c) If one inspects the result in part (b) more closely, it is easy to demonstrate that the expectation value of $\hat{L}_y$ with respect to the state Y_{11} (an eigenstate of $\hat{L}_z$, i.e. a state of definite L_z) vanishes. **Show it**. [Note: No lengthy calculation is needed here.]

[**Important Remarks:** Part (c) shows that a state with definite L_z (namely $+\hbar$) has uncertain values of L_y. In class, we discussed that the z-direction is nothing special. Given that there are 3 possible values of the z-component, there are also 3 possible components in any direction (including y-direction). Y_{11} has definite $L_z = \hbar$. If we take the state and measure L_y, we will obtain different values (sometimes $\hbar$, sometimes 0, sometimes $-\hbar$) when the measurements are carried out on identical copies of the state Y_{11}. The mean value of the result, i.e. the expectation value $\langle L_y \rangle = 0$, as shown in (c).]

SQ26 *2D Isotropic oscillator can also be studied in plane polar coordinates*

In mid-term exam (Question 5), you did a 2D oscillator. You found that when the oscillator is isotropic with identical angular frequency in x and y directions, i.e., $\omega_x = \omega_y = \omega$, degeneracy grows with the allowed value of the energy.

For 2D isotropic oscillator (or circular oscillator), the potential energy function becomes

$$U(x,y) = \frac{1}{2}m\omega^2\left(x^2+y^2\right) = \frac{1}{2}m\omega^2 r^2 = U(r)\ . \tag{2}$$

In 2D, U should be $U(r,\phi)$ in general when expressed in plane polar coordinates. Here, we see that $U(r,\phi) = U(r)$ in the case under consideration. Thus, separation of variables should work and the problem can be done in plane polar coordinates. However, the mathematics in solving the TISE in plane polar coordinates for $U(r)$ given in Eq. (2) is more involved. This is why you were asked to do it in Cartesian coordinates in the mid-term.

TA: Instead of solving TISE, simply take the case of $E = 2\hbar\omega$ as an example. Using $\psi_n(x)$ for the 1D oscillator eigenstate corresponding to the energy E_n, **what are the states** that have the eigenvalue $2\hbar\omega$?

Hence, referring to the explicit form of $\psi_n(x)$, **form two linear combinations** so that the resulting eigenstates have explicitly the form in plane polar coordinates, i.e., $\psi(r,\phi) = R(r)\Phi(\phi)$.

[Remark: One can indeed go further by noticing that $\Phi(\phi) \sim e^{im\phi}$. This is related to SQ24. See also Problem Set 4.]