**PHYS3022 APPLIED QUANTUM MECHANICS**

**SAMPLE QUESTIONS FOR DISCUSSION IN WEEK 13 EXERCISE CLASSES (15 - 18 April 2019)**

The Sample Questions are designed to serve several purposes. They either review what you have learnt in previous courses, supplement our discussions in lectures, or closed related to the questions in an upcoming Problem Set. Students should be able to do the homework problems independently after attending the exercise class. **You should attend one exercise class session.** You are encouraged to think about (or work out) the sample questions before attending exercise class and ask the TA questions.

Progress: We discussed tunnelling and started a crash module on nuclear physics in Week 12.

SQ29 - Single barrier tunnelling with different potential energies on the two sides
SQ30 - Energy to detach a nucleon from a nucleus
SQ31 - Force mediator and its mass in relation to the range of interaction: Klein-Gordon Equation, Yukawa's pion and the Yukawa Potential

SQ29 **Single Barrier Tunnelling with different potential energies on the two sides**

In SQ28 (Week 12), we studied the Probability Current Density for different wavefunctions with travelling waves. Tunnelling is a good place to put them into applications.

Let's consider a single barrier of barrier height $V_0$ and barrier width $L$. Let the barrier be ranged at $0 < x < L$.

In class (notes), we studied the case of $U(x) = 0$ for $x < 0$ and $x > L$, i.e., the potential energy function $U(x)$ is identical on both sides of the barrier. In this case, we write

$$\psi_I(x) = e^{ikx} + r\,e^{-ikx} \quad \text{for } x < 0 \tag{1}$$
$$\psi_{III}(x) = t\,e^{ikx} \quad \text{for } x > L \tag{2}$$

where $r$ is a (generally complex) reflection amplitude and $t$ is a (generally complex) transmission amplitude, and match boundary conditions at $x = 0$ and $x = L$ to obtain $r$ and $t$. In this case, **it so happens** that the **transmission coefficient** $T = |t|^2$ and the **reflection coefficient** $R = |r|^2$ and we can show that $R + T = 1$ (see Problem 7.5).

Here, we consider the case where $U(x) = 0$ for $x < 0$ and $U(x) = -U_0$ for $x > L$, i.e., the potential energy functions are **different**/asymmetric on the two sides. Physically, it is like applying a bias (by a battery say) to help electrons go to the other side.

TA: **Do the right calculation.** For the problem, Eqs. (1) and (2) are modified to become

$$\psi_I(x) = e^{ikx} + r\,e^{-ikx} \quad \text{for } x < 0 \tag{3}$$
$$\psi_{III}(x) = t\,e^{ik'x} \quad \text{for } x > L \tag{4}$$

where $k$ and $k'$ are the wavevectors in region I ($x < 0$) and region III ($x > L$), respectively. **They are different. Defining** the transmission and reflection coefficients properly via the ratios of the probability current densities, **show** that $T \neq |t|^2$ in this case and only using the proper definition will the end result satisfy $R + T = 1$. **Optional:** It will be nice to compare $T(E)$ for the symmetric and antisymmetric cases.

TA: **A careless calculation.** Some students thought that $T = |t|^2$ even in this asymmetric case. **Illustrate** that $R + |t|^2 \neq 1$ for such a wrong definition.

SQ30 **Energy to detach a nucleon from a nucleus.**

Here is a standard example of *energetics* in nuclear physics. [For those preparing for GRE physics test, you should know this.]

**Obtain an expression** for the energy $E$ required to remove the least bound proton from a nucleus $^A_Z X$ by following the steps: (a) Write down the total mass of the products and hence its energy equivalence (usually in atomic masses). (b) Write down the energy of the nucleus to start with (usually in atomic masses). (c) Hence, obtain an expression for the energy $E$ needed to get at the products? [See remarks.]

Hence, **find** the energy $S_p$ required to remove the least tightly bound proton from $^{40}_{20}Ca$.

**Remarks:** The quantity studied here is usually called $S_p$. A same approach can be used to obtain the energy to remove the least bound neutron from a nucleus and it is called $S_n$. Using the independent particle model of nucleus, these quantities are indicative of the well depth. These quantities also show features associated with the magic numbers (see figures in class notes), analogous to the "ionization energy" in atoms. In the Segre Chart, there are proton drip line and neutron drip line beyond which one does not gain any energy by adding one more proton (one more neutron).

SQ31 **Force mediator and its mass in relation to the range of interaction: Klein-Gordon Equation, Yukawa's pion and the Yukawa Potential.**

**Motivation:** In class notes, we made use of the short range and thus the short duration that the particle for mediating the nuclear force exists to estimate the mass of the particle (called Yukawa's $\pi$-meson), in a way similar to the short life time of an atomic excited state. The end result is that **the mass of the mediating particle is inversely proportional to the range** of the interaction. Here, we make an alternative argument based on a relativistic quantum mechanical equation.

**Background:** We know that the Maxwell's equations are at the heart of EM theory. In modern theory, electrons interact by exchanging photons and photons are massless. Therefore we start with EM. In electromagnetism, the scalar potential $\phi$ satisfies a wave equation. In the static case (time derivative term vanishes), the potential everywhere for a point charge at the origin satisfies the Laplace equation $\nabla^2\phi = 0$. (For those who want to put a charge (source) at the origin mathematically, the right-hand side has a delta-function.) The solution is that $\phi(r) \sim 1/r$. It drops gradually and thus EM interaction is referred to as a **long-range interaction**. In a fancy form, we could say that $\phi$ is the bosonic (photon is a boson and it is responsible for EM interaction) potential of a point charge. The photon is massless.

**Start here:** The early success of the Schrödinger Equation in explaining atomic physics (atomic spectrum) had led to attempts in constructing quantum mechanical equations that are consistent with Einstein's special relativity. An early attempt is the Klein-Gordon equation, after the two physicists who worked on it. The idea is: For Schrödinger's non-relativistic QM, we start with the Newtonian expression $E = \frac{p^2}{2m}$ for a free particle. Then we multiply both sides by a wavefunction $\Psi(x,t)$ and so we have $E\Psi = \frac{p^2}{2m}\Psi$. The punch line is that of substituting $E \to i\hbar\frac{\partial}{\partial t}$ and $p \to \frac{\hbar}{i}\frac{\partial}{\partial x}$. Immediately, the Schrödinger Equation for a free particle emerges! It is a handy scheme of going from classical physics to quantum physics! Klein and Gordon, (and Dirac for his Dirac equation of an electron) basically followed the same idea.

**The Klein-Gordon Equation:** Start with the well-known expression of

$$E^2 = m^2c^4 + c^2(p_x^2 + p_y^2 + p_z^2) \tag{5}$$

and follow the same steps, **show** that the **Klein-Gordon equation**

$$-\hbar^2\frac{\partial^2\phi}{\partial t^2} + \hbar^2c^2\nabla^2\phi = m^2c^4\phi. \tag{6}$$

emerges.

[**Remarks on relativistic quantum equations:** (a) In the early years (1926 - 1934), the Klein-Gordon equation was largely ignored because of a "problem" in its probability density. Later on, the equation was found to be the important first step towards a **quantum field theory** for spin-zero bosons by Pauli and Weisskopf (1934). (b) Dirac also started with $E^2 = m^2c^4 + c^2(p_x^2 + p_y^2 + p_z^2)$, but he wanted to write it as $E^2 = (something)^2$ so that the square root can be taken and the problem of having $\partial^2/\partial x^2$ etc. under a square root can be avoided. He got his Dirac Equation, which contains the spin-half of an electron and the anti-particle of an electron (positron) as consequences.]

In nuclear physics, protons and neutrons interact through nuclear force. The key features are that the nuclear force is **short-range** ($\sim 10^{-15}$ m) and strong (of energy scale of $MeV$). In the picture of nuclear force being mediated by the exchange of a particle, which is Yukawa's meson, what the mass of the particle will be. Note that force carriers (the particles responsible for interactions) are **bosons**. The Klein-Gordon equation works for (massive) bosons. So, let's follow the steps in the electrostatic case and obtain the form of the potential. The bosonic (Yukawa's $\pi$-meson or pion) potential now satisfies the **Klein-Gordon equation** as given by Eq. (6). This is analogous to the wave equation satisfied by the potential $\phi_{em}$ in EM. If $m = 0$ (photon, thus EM), then we have

$$\nabla^2\phi_{em} - \frac{1}{c^2}\frac{\partial^2\phi_{em}}{\partial t^2} = 0, \tag{7}$$

which is the EM wave equation for $\phi_{em}$ when the sources are outside the region under consideration. For the static case of Eq. (7), we have the Laplace equation and $\phi_{em} \sim 1/r$.

**Action:** Back to Eq.(6) and consider the static case, we have

$$\nabla^2\phi = \left(\frac{mc}{\hbar}\right)^2 \phi. \tag{8}$$

**Show** that this allows a solution

$$\phi(r) \sim \frac{1}{r}\exp\left(-\frac{mcr}{\hbar}\right) \sim \frac{1}{r}\exp\left(-\frac{r}{R}\right). \tag{9}$$

This bosonic ($\pi$-meson or pion) potential modifies the coulomb form by an exponentially decaying factor. It drops rapidly and it is referred to as a **short range interaction**. Note that there should be no units inside the exponential factor (or sine and cosine). From Eq. (9), **a decaying length $R$ can be identified and it is interpreted as the range**. Therefore, the range is identified to be

$$R = \frac{\hbar}{mc}, \tag{10}$$

which is a result that relates $R$ and the mass of the force carrier $m$.

Finally, **estimate** the mass of Yukawa's $\pi$-meson (the name $\pi$ was given after Yukawa's work) given that the range $R \sim 10^{-15}$ $m$ for nuclear force.

**Remark:** "e" in EM sets the strength of the photon-charge coupling. Now in nuclear force, it is NOT about interaction between charges. Not knowing the detail of the coupling, one can still copy the form of the EM case and introduce a constant $g$ that sets the strength of the pion-nucleon coupling. So, a popular form of the interaction is written as:

$$U(r) \sim \frac{g^2}{r} \exp\left(-\frac{mcr}{\hbar}\right), \tag{11}$$

which is the **Yukawa potential**.

Yukawa focused on the nuclear force and he had no idea that there are deeper structures (quarks) inside protons and neutrons. However, Eq. (10) provides a way to estimate the mass of force carriers of other interactions. Let's say there is an even shorter range interaction of range $10^{-18}$ $m$, **what** is the mass of the carrier?