Appendix: Low temperature C_V of Ideal Fermi Gas

From statistical physics, the Fermi-Dirac distribution

$$\frac{1}{e^{\beta(\varepsilon-\mu)}+1}$$

gives the number of particles[+] per single-particle state in a system at equilibrium at a temperature T.

μ = chemical potential
ε = energy of single-particle state
$\beta \equiv \frac{1}{kT}$.

$g(\varepsilon)d\varepsilon$ = number of single-particle states with energy between ε to $\varepsilon + d\varepsilon$

Let's consider an ideal fermi gas with N particles in Volume V.

Thus,
$$N = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty \frac{\varepsilon^{1/2}}{e^{\beta(\varepsilon-\mu)}+1}\,d\varepsilon \qquad \text{(A1)}$$

$$U = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty \frac{\varepsilon^{3/2}}{e^{\beta(\varepsilon-\mu)}+1}\,d\varepsilon \qquad \text{(A2)}$$

Eqs. (A1) and (A2) govern the physics. They work at any temperature.

[+] Since this number is between 0 and 1, an alternative interpretation is that it is the probability of a single-particle state at energy ε being occupied.

- Look at Eq. (A1), it is an implicit equation to determine $\mu(T)$.
- With $\mu(T)$, Eq. (A2) gives $U(T)$ and $C_V = \frac{\partial U}{\partial T}$.
- From $T=0$ physics, $E_F = \mu(T=0) \sim 3\text{-}5\,eV$ for metals.

Even for room temperature $T \sim 300K \sim 0.024\,eV$

$$kT \ll E_F$$

Thus, we expect $\mu(T\sim 300K) \approx E_F$

$\Rightarrow$ shift of μ with temperature is small compared with E_F

With this, Eqs. (A1) and (A2) amount to doing an integral of the form

$$\int_0^\infty \frac{f(\varepsilon)\,d\varepsilon}{e^{\beta(\varepsilon-\mu)}+1} \quad \text{for } kT \ll \mu$$

Sommerfeld formula:

$$\int_0^\infty \frac{f(\varepsilon)}{e^{\beta(\varepsilon-\mu)}+1}\,d\varepsilon \approx \int_0^\mu f(\varepsilon)\,d\varepsilon + \frac{\pi^2}{6}(kT)^2 f'(\mu) + \cdots \quad (\beta\mu \gg 1) \qquad \text{(A3)}$$

where $f'(\mu) = \left.\frac{df}{d\varepsilon}\right|_{\varepsilon=\mu}$

In Eqs. (A1) and (A2), there is a prefactor $\frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \equiv \mathcal{A}$.

- Applying Eq. (A3) to Eq. (A1), we have at low temperatures:

$$N = \mathcal{A}\int_0^\infty \frac{\varepsilon^{1/2}}{e^{\beta(\varepsilon-\mu)}+1}\,d\varepsilon = \mathcal{A}\left[\frac{2}{3}\mu^{3/2} + \frac{\pi^2}{12}(kT)^2\mu^{-1/2} + \cdots\right] \quad (A4)$$

- Since Eq. (A1) is good at any temperature, it is good at $T=0$.
  At $T=0$, it becomes:

$$N = \mathcal{A}\int_0^{E_F} \varepsilon^{1/2}\,d\varepsilon = \frac{2}{3}\mathcal{A}E_F^{3/2} \quad (A5)$$

But N is N! Thus, Eqs. (A4) and (A5) give

$$\frac{2}{3}\mathcal{A}E_F^{3/2} = \mathcal{A}\left[\frac{2}{3}\mu^{3/2} + \frac{\pi^2}{12}(kT)^2\mu^{-1/2} + \cdots\right]$$

$$\Rightarrow E_F^{3/2} = \mu^{3/2}\left[1 + \frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2 + \cdots\right] \qquad \text{Note: } \frac{kT}{\mu} \ll 1$$

$$\therefore \mu(T) = E_F\left[1 + \frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2 + \cdots\right]^{-2/3}$$

$$\approx E_F\left(1 - \frac{\pi^2}{12}\left(\frac{kT}{\mu}\right)^2\right)$$

$$\Rightarrow \mu(T) \approx E_F\left(1 - \frac{\pi^2}{12}\left(\frac{kT}{E_F}\right)^2\right) \quad (A6)$$

- Eq. (A1) determines $\mu(T)$
- $\mu(T)$ shifts with T
- But the shift is tiny as $\left(\frac{kT}{E_F}\right)^2 \ll 1$
- shifts towards the side with smaller density of states

- Next, we use Eq. (A2) to obtain $U(T)$.

$$U = \mathcal{A}\int_0^\infty \frac{\varepsilon^{3/2}}{e^{\beta(\varepsilon-\mu)}+1}\,d\varepsilon$$

$$= \mathcal{A}\left[\frac{2}{5}\mu^{5/2} + \frac{\pi^2}{6}(kT)^2\frac{3}{2}\mu^{1/2} + \cdots\right] \quad \text{using (A3)}$$

$$= \mathcal{A}\frac{2}{5}\mu^{5/2}\left[1 + \frac{5\pi^2}{8}\left(\frac{kT}{\mu}\right)^2 + \cdots\right]$$

$$= \frac{3}{5}\cdot\underbrace{\frac{2}{3}\mathcal{A}E_F^{3/2}}_{N \text{ (see (A5))}}E_F\underbrace{\left(\frac{\mu}{E_F}\right)^{5/2}}_{\text{use (A6)}}\left[1 + \frac{5\pi^2}{8}\left(\frac{kT}{\mu}\right)^2\right]$$

$$\cong \frac{3}{5}NE_F\left(1 - \frac{\pi^2}{12}\left(\frac{kT}{E_F}\right)^2\right)^{5/2}\left(1 + \frac{5\pi^2}{8}\left(\frac{kT}{E_F}\right)^2\right)$$

keep order of $\left(\frac{kT}{E_F}\right)^2$

$$\approx \frac{3}{5}NE_F\left(1 - \frac{5\pi^2}{24}\left(\frac{kT}{E_F}\right)^2\right)\left(1 + \frac{5\pi^2}{8}\left(\frac{kT}{E_F}\right)^2\right)$$

$$\approx \frac{3}{5}NE_F\left(1 + \frac{5\pi^2}{12}\left(\frac{kT}{E_F}\right)^2\right)$$

$$= \underbrace{\frac{3}{5}NE_F}_{U(T=0)} + \underbrace{\frac{\pi^2}{4}NE_F\left(\frac{kT}{E_F}\right)^2}_{\text{first term goes like } T^2 \text{ and thus } C_V \sim T}$$

$$\text{Eq. (A5)} \Rightarrow N = \frac{2}{3}\mathcal{A}E_F^{3/2} = \frac{2}{3}\left(\mathcal{A}E_F^{1/2}\right)E_F = \frac{2}{3}\underset{\text{DOS at } E=E_F}{g(E_F)}E_F$$

$$\therefore U(T) = U(T=0) + \frac{\pi^2}{6}g(E_F)(kT)^2$$

$$C_V = \frac{\pi^2}{3}g(E_F)(kT)k \ \#$$

Using hand-waving argument, $C_V = 2g(E_F)(kT)\cdot k$ (not bad!)