PHYS4031 STATISTICAL MECHANICS

SAMPLE QUESTION FOR DISCUSSION IN WEEK 10 EXERCISE CLASSES (7, 9 November 2016)

You may want to think about them before attending exercise class.
SQ23: Critical behavior as predicted by the van der Waals equation
SQ24: Taylor expansions, behavior of $\sinh x$, $\cosh x$ and $\tanh x$ for small x, and mean field equation
SQ25: Magnetic susceptibility for $T \to T_c^{\pm}$ within mean field theory
Optional questions for TA on SQ23-25 collectively

About SQ23 and SQ24: They serve to illustrate the similarities between the critical behavior in a vapor-liquid transition as predicted by the van der Waals equation and in a paramagnetic-ferromagnetic transition as predicted by the Ising model within the mean field theory.

SQ23 **Critical behavior as predicted by van der Waals equation**

The van der Waals equation of state in reduced variables is particularly simple:

$$\left(p_R + \frac{3}{v_R^2}\right)\left(v_R - \frac{1}{3}\right) = \frac{8}{3}\, T_R \,. \tag{1}$$

In this form, the critical point (C.P.) is at $(p_R = 1, v_R = 1, T_R = 1)$. The critical behavior refers to how the system behaves **in the vicinity** of the critical point.

TA: The task is to work out what the equation says about the behavior near the critical point. Starting with tiny deviations from the C.P., i.e.,

$$p_R = 1 + \pi \,, \qquad v_R = 1 + \psi \,, \qquad T_R = 1 + t \tag{2}$$

where π, ψ, and t can be positive or negative but they satisfy $|\pi| \ll 1$, $|\psi| \ll 1$ and $|t| \ll 1$, explore how ψ depends on $|t|$ for $t < 0$ (but $|t| \ll 1$) and identify the critical exponent β, and the functional form of the critical isotherm (meaning right at $T = T_c$ and thus $T_R = 1$ and $t = 0$) near C.P., i.e., how π goes with ψ at $t = 0$ and identify the exponent δ.

[Remark: Although we are using van der Waals equation here, the procedure of extracting the critical behavior is general.]

SQ24 **$\sinh x$, $\cosh x$, $\tanh x$ for small x and mean field theory of Ising model**

There are many $\tanh x$, $\cosh x$, $\sinh x$ or even *sech* x (what is it?) floating around in mean field theory calculations. Usually, x is related to the (dimensionless) magnetization per spin m and thus it is small near the critical point. As such we care about how these functions behave for small arguments.

(a) TA: Remind students how $\sinh x$, $\cosh x$ and $\cosh^2 x$ behave for small x. Hence, illustrate how one can obtain the behavior of $\tanh x$ at small x from the known behavior of $\sinh x$ and $\cosh x$. [Remarks: These are essential for manipulation the mean field equation and extract critical behavior. In addition, Taylor expansion will also be important in studying phenomena near the critical point.]

(b) The most important equation in the mean field approximation to the Ising model is the self-consistent equation that determines m as a function of the temperature T and the applied field B, i.e., $m(T, B)$. The mean field equation reads

$$m = \tanh\left(\frac{Jz}{kT} m + \frac{B}{kT}\right) \,. \tag{3}$$

Here, m is a dimensionless magnetization per moment (per spin) with value between $-1 \le m \le 1$. The critical point of the problem is $(T = T_c, B = 0)$.

TA: Explore how $m(T, 0)$ behaves near T_c, extract the critical behavior for $T \to T_c^-$ (T approach T_c from below) and identify the critical exponent β. Similarly, consider how m and B are related near C.P. on the critical isotherm, i.e. $T = T_c$. Compare the values of the two exponents here with those in SQ23.

[Remarks: Although we will discuss the critical behavior of the Ising model in class, here SQ23 and SQ24 serve to illustrate some common features of critical behavior as predicted by mean field theories. We will need the result in SQ24 to move onto SQ25.]

SQ25 **Magnetic susceptibility as $T \to T_c^{\pm}$ in mean field theory.**

We will show that the susceptibility χ as $T \to T_c^+$ in the paramagnetic phase diverges as $1/(T - T_c)$, in agreement with experimental observations. Here, TA will work out the case for $T \to T_c$ from below.

Think about the physics. The susceptibility $\chi(T, B = 0)$ is formally defined as $\lim_{B\to 0}(\partial m/\partial B)_T$. For $T \gg T_c$, the kT effect dominates and it tends to randomize the spins. The system is in the paramagnetic phase with $kT \gg B$ (recall that B is an energy characterizing the Zeeman splitting) and thus the magnetic susceptibility is small ($\chi \sim 1/T$ as $T \gg T_c$). In the opposite limit of $T \to 0$, the spin-spin interaction wins and spins tend to be aligned, even in the absence of B. At low temperatures, the spins are almost aligned. Applying a small field B will only lead to a *slightly* better alignment (assuming the field is in the direction of the alignment). As such, χ is also small. In fact, χ increases as we approach T_c both from above and from below. **The divergence of a response function, χ in this case, is often an important indicator of critical phenomena.** Here, TA will work it out in both directions.

(a) Let's get an expression for the magnetic susceptibility formally from the mean field equation. Taking a derivative of the mean field equation w.r.t. B at constant T, show that

$$\chi(T, B = 0) = \lim_{B\to 0}\left(\frac{\partial m}{\partial B}\right)_T = \frac{1}{k}\frac{1}{\left(T\cosh^2[\frac{T_c}{T}m(T, 0)] - T_c\right)} \tag{4}$$

where $m(T, 0)$ is how m depends on T in the absence of B as obtained in SQ24.

(b) (Easier part) Now consider $T > T_c$ and $T \to T_c$ from above. Obtain the behavior of $\chi(T, 0)$ as $T \to T_c^+$ from the result in part (a). The exponent is called γ^+ with the superscript indicating T_c is approach from above.

(c) (Trickier part) Now consider $T < T_c$ and $T \to T_c$ from below. Using the behavior of $\cosh^2 x$ and $m(T, 0)$ as obtained in SQ24, show that χ also diverges as $T \to T_c^-$ with an exponent $\gamma^- = \gamma^+$ BUT with a prefactor that is different from part (b) by a factor of 1/2. Hence, sketch $\chi(T, 0)$ as a function of T. [Students: Note that Taylor expansion is something easy to do, but one needs to do it carefully.]

[Remarks: At this point, SQ24 and SQ25 give the mean field predicted behavior of $m \sim (T_c - T)^{\beta}$ ($\beta = 1/2$), $m \sim B^{1/\delta}$ ($\delta = 3$) and $\chi \sim |T - T_c|^{-\gamma}$ of the Ising model near the C.P.]

Optional questions on SQ23,23,25 for the TA:

(i) We discussed the susceptibility in SQ25 for the Ising model. This is natural as the problem is about magnetic properties. In the vapor-liquid case, is there an analogous behavior related to a physical quantity?

(ii) In these SQ's, the β and δ exponents are found to be the same when we invoke the mean field theory for the Ising model and the van der Waals equation for the interacting gas. In the Ising mean field theory, we found a way to turn an interacting spin problem into an effective independent spin problem. Is there a way to see that in the steps of obtaining the van der Waals equation, we also turned an interacting particle problem into an effective non-interacting particle problem?