Damped harmonic motion

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Harmonic motion is studied in the presence of a damping force proportional to the velocity. The complex method is introduced, and the different cases of under-damping, over-damping and critical damping are analyzed.

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1 Introduction and simple analysis

1.1 The model

A physical oscillator (e.g., a mass m tied to a spring with force constant k) after being set into motion would gradually lose energy, and the amplitude of motion would decrease with time. The model of simple harmonic motion (SHM) is obviously inadequate for such a situation. The loss of energy can be modelled by the addition of a damping or viscous force F_v that is opposite to the velocity, and has a magnitude proportional to the velocity $v = \dot{x}$:

$$F_v = -b\frac{dx}{dt} \tag{1}$$

so that the equation of motion is

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}$$

or

$$\left(\frac{d^2}{dt^2} + \alpha\frac{d}{dt} + \omega_0^2\right)x(t) = 0 \tag{2}$$

where

$$\alpha = \frac{b}{m} \quad , \quad \omega_0^2 = \frac{k}{m} \tag{3}$$

The special case $\alpha = 0$ reduces to SHM. This module studies motion described by (2).

1.2 Numerical solution

A numerical solution is shown in the spreadsheet `eqm-d1.xls`. The sheet gives the motion for $\alpha = 0.3$, $\omega_0^2 = 4.0$ and initial conditions $x(0) = 1.0$, $v(0) = 0.0$. A time step of $\Delta t = 0.05$ is used. It is found that the next maximum occurs at $t \approx 3.15$ with amplitude ≈ 0.73. All the input parameters can be easily varied. Students should play with the spreadsheet to get a sense of how the solution depends on the system parameters.

1.3 Guessing a solution

This Section takes a physical and heuristic approach, and *guesses* the general solution to (2). By the general theorems on ODEs and indeed as shown by the numerical method (see above), there is one and only one solution if two initial conditions are given. If the conjectured solution (a) is checked to satisfy (2) and (b) contains two free parameters,[1] then it must be the correct general solution. With this in mind, we proceed to make a conjecture, based on physical intuition.

Recall that without damping, i.e., if $\alpha = 0$, then the general solution is

$$x(t) \quad = \quad A\cos(\omega_0 t + \phi_0) \tag{4}$$

The two arbitrary constants A and ϕ_0 (or equivalently two *linear* free parameters B and C related to $A\cos\phi_0$ and $A\sin\phi_0$) are determined by the initial conditions.

Now we conjecture that there are two changes.

- The amplitude A, instead of being constant, now decreases with time in some fashion

$$A \quad \mapsto \quad A(t)$$

  where $A(t)$ decreases to zero at a rate determined by α.

- The damping force would cause the periodic motion to be *slower* or the period to be longer. So we guess that in the argument of the cosine

$$\omega_0 t \quad \mapsto \quad \Omega t$$

  where Ω is smaller than ω_0 by an amount determined by α.

Exponential decay

For any *linear* system, the only possible decay is an exponential one. Let us first phrase the argument physically.

Imagine $A(t)$ dropping (a) from 1 to 1/2, and (b) from 1/2 to 1/4. The latter is just the former multiplied by an overall factor of 1/2 — for a linear system, a multiple of a solution is also a solution. So the time taken for the two processes are identical; let us call it the *half-life* T.[2] Thus

- In a time $t = T$, $A(t)$ decreases by 1/2.
- In a time $t = 2T$, $A(t)$ decreases by $(1/2)^2$.
- In a time $t = nT$, $A(t)$ decreases by $(1/2)^n = (1/2)^{t/T}$.

[1] To be a bit more careful, we need to check that the two parameters can be converted to two *linear* parameters.

[2] Not to be confused with the period. The half-life will not be mentioned after we introduce τ below.

So the functional form must be

$$A(t) \quad = \quad A\left(\frac{1}{2}\right)^{t/T} \tag{5}$$

where the prefactor A is the initial value. This idea of decaying by geometric ratios and the concept of half-life should be familiar from radioactive decay. Our emphasis is that it simply follows from linearity, or the fact that the overall scale cannot matter.

The formula (5) is conventionally written in a different but equivalent way. Start with

$$\begin{aligned} 2 &= e^{a} \\ 1/2 &= e^{-a} \\ (1/2)^{t/T} &= \left(e^{-a}\right)^{t/T} = e^{-at/T} \end{aligned}$$

Thus

$$A(t) \quad = \quad A\,e^{-at/T} \tag{6}$$

Note that $a = \ln 2 = 0.693$. Introduce another time scale τ and its inverse γ by

$$\begin{aligned} \tau &= \frac{T}{a} = 1.44\,T \\ \gamma &= \frac{1}{\tau} \end{aligned}$$

then (6) can be re-written as

$$A(t) \quad = \quad A\,e^{-\gamma t} = A\,e^{-t/\tau} \tag{7}$$

Over one *characteristic time* τ the amplitude decays by a factor of e; γ is the decay rate. The expression (7) is more convenient when we want to differentiate.

A more formal derivation is to note that $A(t)$ must decrease by a definite *fraction* per unit time — linearity means absolute scale does not matter, so only the fraction can enter. Thus

$$\frac{dA(t)}{dt} \quad = \quad -\,\gamma\,A(t)$$

from which (7) follows trivially.

Form of solution

Thus the conjectured solution is of the form

$$x(t) \quad = \quad A\,e^{-\gamma t}\cos(\Omega t + \phi_0) \tag{8}$$

There are four parameters.

- The parameters γ and Ω are determined by the ODE.

- The parameters A and ϕ_0 are determined by initial conditions.

The behavior of the conjectured solution is shown in **Figure 1**, for the case where γ is small, i.e., the amplitude decreases only slightly in each period. The dotted lines are the *envelope*

$$x_e(t) \quad = \quad \pm A\, e^{-\gamma t} \tag{9}$$

and the solid line is (8). It touches the upper (lower) dotted line when the cosine reaches $+1$ (-1).

By the way, the time t_n when the cosine is $+1$ and the time t'_n when the displacement is maximum are not exactly the same; see **Figure 2**.

Problem 1
For simplicity let $\phi_0 = 0$. Let $T = 2\pi/\Omega$.
(a) Show that the cosine has the value $+1$ whenever $t = t_n = nT$.
(b) Show that the displacement $x(t)$ attains its local maxima whenever $t = t'_n = nT - \Delta t$, where Δt is a constant (independent of n). Also determine Δt. §

Period
At least for weak damping, one still talks about the period T (as above), which is the interval between successive local maxima, even though the motion is not strictly periodic.

1.4 Checking the solution

Next check the solution and determine γ and Ω. We can reduce the apparent complexity and gain clarity by a couple of tricks.

- The overall amplitude A does not matter; it will appear in every term in the ODE (2) and can be cancelled. So we may as well save some writing and set $A = 1$ in the derivation.

- No matter how many times we differentiate, there will be only two types of terms

$$\begin{aligned} e^{-\gamma t} \cos(\Omega t + \phi_0) &= e^{-\gamma t}\, C \\ e^{-\gamma t} \sin(\Omega t + \phi_0) &= e^{-\gamma t}\, S \end{aligned}$$

- In fact the displacement, velocity and acceleration must take the form

$$\begin{aligned} x(t) &= e^{-\gamma t}\,(p_0 C + q_0 S) \\ \dot{x}(t) &= e^{-\gamma t}\,(p_1 C + q_1 S) \\ \ddot{x}(t) &= e^{-\gamma t}\,(p_2 C + q_2 S) \end{aligned}$$

where for the assumed solution (8)

$$p_0 = 1 \qquad , \qquad q_0 = 0 \tag{10}$$

- When these are put into (2), the conditions will be

$$\begin{aligned} &\left(p_2 + \alpha p_1 + \omega_0^2 p_0\right) e^{-\gamma t}\, C \\ &+ \left(q_2 + \alpha q_1 + \omega_0^2 q_0\right) e^{-\gamma t}\, S = 0 \end{aligned}$$

Since this must hold as an identity in t, the two terms must separately vanish, and we get two conditions on the coefficients:

$$\begin{aligned} p_2 + \alpha p_1 + \omega_0^2 p_0 &= 0 \\ q_2 + \alpha q_1 + \omega_0^2 q_0 &= 0 \end{aligned} \tag{11}$$

The above sets out the schema for the derivation, and the actual evaluation is left as an exercise.

Problem 2
Starting with (10), calculate p_1, q_1 and p_2, q_2. Hence write out (11) explicitly, and show that the solution is

$$\begin{aligned} \gamma &= \alpha/2 \\ \Omega &= \sqrt{\omega_0^2 - \gamma^2} \end{aligned} \tag{12}$$

Note that these make sense: (a) the rate of exponential decrease, γ, is proportional to the amount of damping α; and (b) the frequency of motion Ω is reduced from ω_0 by an amount related to the damping. §

Problem 3
Consider a system with $\alpha = 2$, $\omega_0 = 3$, and initial conditions $x(0) = 1$, $v(0) = 0$. Find the position x at time $t = 0.5$
(a) by making use of the analytic solution, and
(b) numerically using a spreadsheet, and changing the time step until the result is reasonably accurate. §

2 Complex method

Motivation
The complex method is now introduced. There are at least two motivations.

- The calculation will be actually simpler. Instead of keeping track of *two* types of terms, there will be only *one* type.
- From (12) one would naturally ask what happens when $\gamma > \omega_0$ — would not the square root become imaginary?

The main elements of the method have already been outlined in the module on solving ODEs, but a self-contained derivation is nevertheless presented below.

Complex solution
Start by looking for a complex solution $\tilde{x}(t)$ satisfying

$$\left(\frac{d^2}{dt^2} + \alpha\frac{d}{dt} + \omega_0^2\right)\tilde{x}(t) = 0 \qquad (13)$$

We conjecture a solution

$$\tilde{x}(t) = \tilde{A}\,e^{i\omega t} \qquad (14)$$

where $\tilde{A}$ is a complex constant. The form of (14) ensures that

$$\frac{d}{dt} \mapsto i\omega$$

so (13) becomes an *algebraic* equation

$$(i\omega)^2 + \alpha(i\omega) + \omega_0^2 = 0 \qquad (15)$$

This is a quadratic equation for ω, for which the solution is

$$\omega = i\gamma \pm \Omega \qquad (16)$$

where γ and Ω are given in (12). For the moment suppose the damping is small and Ω is real.

Problem 4
Check the solution (16) by solving the quadratic in (15). §

Take the real part
Henceforth take the + sign. (See Problem 5 below.) Then

$$e^{i\omega t} = e^{-\gamma t}\,e^{i\Omega t}$$

Also represent the complex amplitude as

$$\tilde{A} = A\,e^{i\phi_0}$$

Then

$$\tilde{x}(t) = A\,e^{-\gamma t}\,e^{i(\Omega t+\phi_0)} \qquad (17)$$

Now if we write

$$\tilde{x}(t) = x(t) + iy(t)$$

then the real part $x(t)$ by itself is a solution to the ODE (as is the imaginary part $y(t)$ by itself). Applying this idea to (17), we find

$$x(t) = A\,e^{-\gamma t}\,\cos(\Omega t + \phi_0)$$

recovering the solution as before.

Problem 5
Take the minus sign in (16) and show that there is no new solution. §

3 Different cases of damping

From the last Section we found that

$$\begin{aligned} x(t) &= \Re\,\tilde{x}(t) \\ \tilde{x}(t) &= \tilde{A}\,e^{i\omega t} \\ \omega &= i\alpha/2 \pm \sqrt{\omega_0^2 - \alpha^2/4} \qquad (18) \end{aligned}$$

and the analysis in the last Section implicitly assumed that the discriminant

$$\Delta = \omega_0^2 - \alpha^2/4$$

is positive, as would be the case if the damping coefficient α is small. Evidently this is only one of three possible cases, which we now analyze more systematically.

3.1 Under-damped

If $\Delta > 0$, then Ω as defined is real, then

$$\begin{aligned} \tilde{x}(t) &= A\,e^{i\phi_0}\cdot e^{-\gamma t}\,e^{i\Omega t} \\ &= A\,e^{-\gamma t}\cdot e^{i(\Omega t+\phi_0)} \\ x(t) &= A\,e^{-\gamma t}\cdot\cos(\Omega t + \phi_0) \qquad (19) \end{aligned}$$

as discussed. This is an *oscillatory* solution, since the cosine alternates in sign an infinite number of times (**Figure 1**). This case is the most intuitively obvious, since it is "closest" to the undamped case with $\alpha = 0$.

3.2 Over-damped

If $\Delta < 0$, then both roots for ω in (18) are imaginary, and we denote them as $-i\gamma_{1,2}$, where

$$\begin{aligned} \gamma_1 &= \frac{\alpha}{2} + \frac{1}{2}\sqrt{\alpha^2 - 4\omega_0^2} \\ \gamma_2 &= \frac{\alpha}{2} - \frac{1}{2}\sqrt{\alpha^2 - 4\omega_0^2} \end{aligned} \tag{20}$$

It is easy to see that

$$\gamma_1 > \gamma_2 > 0$$

(The label of 1, 2 is only a matter of convention.)

The general solution, with two arbitrary constants, is given by

$$x(t) = A_1 e^{-\gamma_1 t} + A_2 e^{-\gamma_2 t} \tag{21}$$

For $t > 0$, this may either have only one sign (i.e., never crossing zero) or it may cross zero once.

Problem 6
Under what conditions on the parameters γ_j and A_j would there be a zero-crossing for $t > 0$? §

Problem 7
Consider a system with $\alpha = 4$, $\omega_0 = 1$, and initial condition $x(0) = 1$, $v(0) = 0$. Find the position $x(t)$ when $t = 0.3$. Also check against the numerical solution using a suitably small time step. §

3.3 Critically damped

Number of solutions
Next consider the case in between: what happens if $\Delta = 0$? The quadratic equation (15) has only *one* root, but we know that (e.g., from the need to match initial conditions) that there must be two independent solutions. So what is wrong?

Only one thing is wrong: the assumption that there are two solutions of an exponent form, e.g., $e^{i\omega t}$, for some complex ω. There is no general theorem that says this must be the case. There must be another solution that is *not* of this form.

To get at the other solution let us write the basic ODE in this case as

$$\left(\frac{d}{dt} + \gamma\right)^2 x(t) = 0 \tag{22}$$

This representation follows the same idea as the characteristic polynomial having two identical factors.

Guessing the other solution
We can simply guess and check the solutions.

Problem 8
Check that the following are two solutions:

$$\begin{aligned} x(t) &= e^{-\gamma t} \\ x(t) &= t\, e^{-\gamma t} \end{aligned} \tag{23}$$

by direct substitution into (22). §

Problem 9
Find the independent solutions to

$$\left(\frac{d}{dt} + \gamma\right)^3 x(t) = 0$$

and see whether you can conjecture and even prove a more general statement. §

The general problem of multiple merged roots in the characteristic polynomial of a dissipative system is of some interest, even recently.[3]

General solution
The general solution is therefore

$$x(t) = (A + Bt)\, e^{-\gamma t}$$

By taking a limit*
**This part is more advanced and can be skipped.*

A systematic way to understand the second solution is to approaching the critical case as a limit — which will establish the link between the three cases, despite the apparent "discontinuity" in the analytic form. So consider the problem

$$\left(\frac{d}{dt} + \gamma\right)\left(\frac{d}{dt} + \gamma + \epsilon\right) x(t) = 0 \tag{24}$$

subject to the initial condition $x(0) = 1$, $\dot{x}(0) = 0$, in the limit $\epsilon \to 0$. (In general, other initial conditions will also reveal the same conclusion.)

For $\epsilon \neq 0$, this is not the critical case, and the general solution is

$$x(t) = A\, e^{-\gamma t} + B\, e^{-(\gamma+\epsilon)t} \tag{25}$$

[3] See e.g., A van den Brink and K Young, "Jordan blocks and generalized bi-orthogonal bases: realizations in open wave systems", Journal of Physics A: Mathematical and General, **34**, 2607 (2001). DOI: 10.1088/0305-4470/34/12/308. You may be surprised that even in the 21st century there was something slightly novel to discover in such classical problems.

Problem 10
Apply the initial conditions and show

$$A = 1 + \frac{\gamma}{\epsilon} \qquad , \quad B = -\frac{\gamma}{\epsilon} \tag{26}$$

Then write out (25) in the limit $\epsilon \to 0$. Hint: Expand $e^{-\epsilon t}$ in powers of ϵ. §

This approach makes it clear why a power of t appears: It comes from ϵt in the expansion of $e^{-\epsilon t}$. The factor of ϵ cancels $1/\epsilon$ in the coefficients in (26). Higher powers would give $\epsilon^{-1} \cdot (\epsilon t)^n$ with $n \geq 2$, and therefore vanish when $\epsilon \to 0$.

4 Energy

General remarks
In the presence of a damping force, the total energy would gradually be lost. This Section records how the total energy decreases with time. For simplicity only the under-damped case, in particular the case with $\gamma \ll \Omega$, will be shown explicitly. Again, to avoid distraction by the mathematics, we state the result first (**Figure 3**):

- The total energy $\mathcal{E}$ decreases as $\exp(-2\gamma t)$, i.e., like the *square* of the amplitude.
- But superimposed on this smooth decrease are small-amplitude oscillations (also decreasing with the same exponent) that vary at the second harmonic 2Ω. These oscillations have approximately zero average.

Displacement and velocity
For the under-damped case, the displacement is taken to be

$$x = A\, e^{-\gamma t} \cos \Omega t$$

An arbitrary initial phase ϕ_0 can be added to the argument of the cosine without affecting the analysis below. The velcoity is therefore

$$v = A\, e^{-\gamma t} \left(-\gamma \cos \Omega t - \Omega \sin \Omega t \right)$$

in which the first term comes from differentiating the exponential and the second come from differentiating the cosine.

Potential energy

$$\begin{aligned} U(t) &= \frac{1}{2} k x(t)^2 \\ &= \frac{1}{2} k A^2\, e^{-2\gamma t} \cos^2 \Omega t \\ &= \frac{1}{4} k A^2\, e^{-2\gamma t} \left(1 + \cos 2\Omega t\right) \end{aligned} \tag{27}$$

Kinetic energy

$$\begin{aligned} K(t) &= \frac{1}{2} m v(t)^2 \\ &= \frac{1}{2} m A^2\, e^{-2\gamma t} \big(\gamma^2 \cos^2 \Omega t + \Omega^2 \sin^2 \Omega t \\ &\qquad + 2\gamma\Omega \cos \Omega t \sin \Omega t\big) \\ &= \frac{1}{4} m A^2 e^{-2\gamma t} \big[\gamma^2 (1 + \cos 2\Omega t) \\ &\qquad + \Omega^2 (1 - \cos 2\Omega t) + 2\gamma\Omega \sin 2\Omega t\big] \end{aligned} \tag{28}$$

Total energy
Consider the total energy $\mathcal{E} = U + K$ and its non-oscillating part $\bar{\mathcal{E}}$ obtained by dropping the terms that oscillate at the second harmonic. Then

$$\begin{aligned} \bar{\mathcal{E}} &= \frac{1}{4} \left[k + m(\gamma^2 + \Omega^2)\right] A^2\, e^{-2\gamma t} \\ &= \frac{1}{2} k A^2\, e^{-2\gamma t} \end{aligned} \tag{29}$$

since $\gamma^2 + \Omega^2 = \omega_0^2$ and $m\omega_0^2 = k$.

Problem 11
The remaining terms are of course oscillatory. But in addition, show that their amplitudes go as γ or γ^2. Thus, for weak damping, these are also small. Hint: Eliminate Ω from the prefactors and express in terms of ω_0 and γ. §

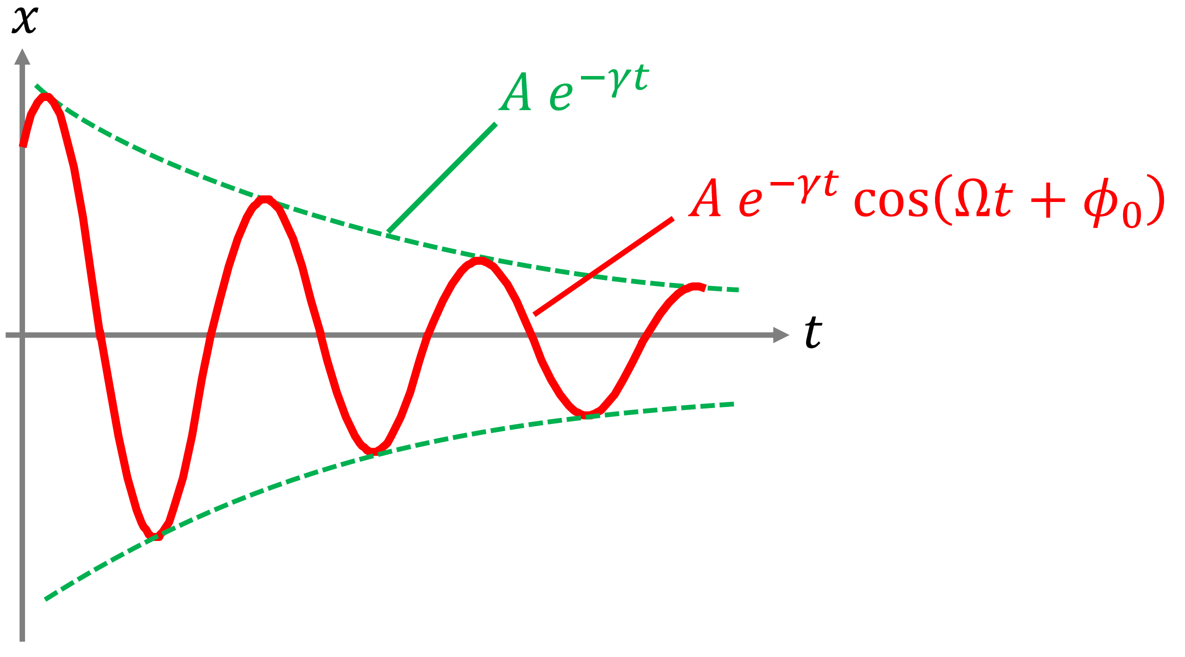

Figure 1

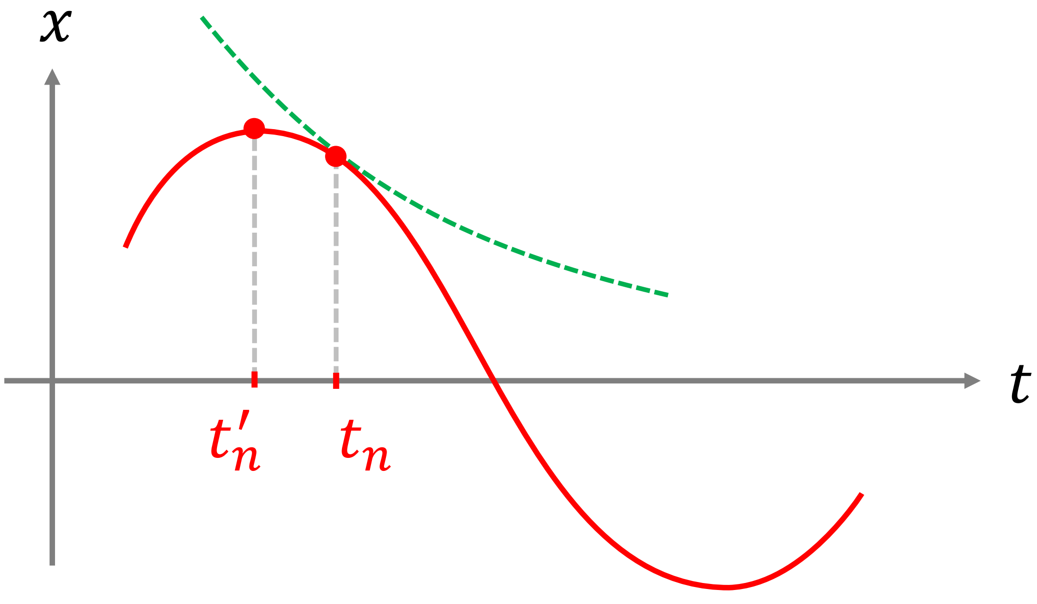

Figure 2

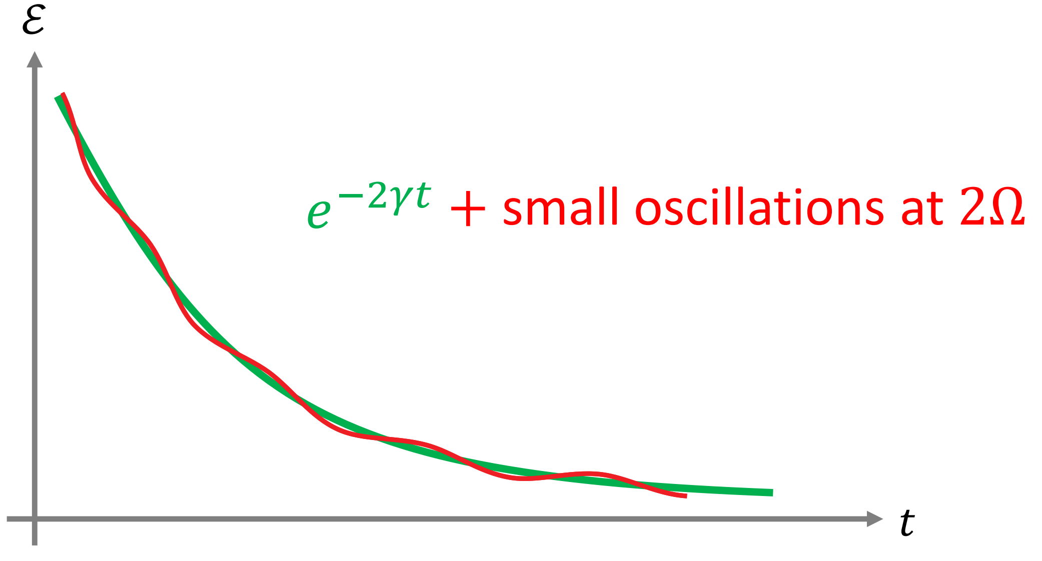

Figure 3