Rolling

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Rolling is studied, as an example of motions that combine translation with rotation. Friction and the condition of no slipping are important concepts.

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1 Introduction

1.1 Translation and rotation

The *configuration* of a rigid body can be described in two steps.

- Choose a point C on the body and specify its displacement $\vec{r}$ from the origin O. Here we take $\vec{r}$ to be in the x–y plane: $\vec{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$.
- Then a number of angles are used to specify the orientation of the body about C, relative to a fixed set of axes. Here we assume the orientation involves only one angle of rotation ϕ about the z-axis.

The situation is illustrated in **Figure 1**.[1]

The *motion* is then given by how $\vec{r}(t)$ and $\phi(t)$ change with time. The former describes the motion of a hypothetical point particle, and is called *translation*; the latter is a *rotation*. This module studies the combination of these two components of motion in simple cases.

For describing the configuration and its change (i.e., kinematics), C can be any point on the body. But for analyzing forces and torques (i.e., dynamics), it is often best to choose C to be the CM — as we shall see below.

1.2 Examples

It is useful to start with some examples to illustrate the issues that may arise.

Car
Consider a car moving on a horizontal road. The center C of each tyre undergoes linear motion, and the tyre rotates about C, the axle of the wheel (**Figure 2**).

Suppose the car accelerates. The only horizontal force acting on the system is friction due to the road. Thus we find that *friction is involved.* The

[1] Thus in this restricted scenario, the kinematics will be described by three variables: two Cartesian coordinates and one angle. The most general case for a rigid body is described by six variables: three Cartesian coordinates and three angles.

frictional force is in the forward direction, by Newton's second law. You may find this perplexing; we shall explain later.

"Normally" the linear motion of the car and the rotation of the wheel are related: every time the wheel rotates once, by an angle 2π, the car moves forward by $2\pi R$, where R is the radius of the wheel. But if the driver steps too hard on the brake, the rotational motion of the wheel comes (almost) to a stop, but the car continues to move forward — the car skids; or we say the tyres slip. In this case, the rotation of the wheel and the linear motion no longer have a definite relationship. This example shows that we must distinguish between rolling without slipping (the "normal" case) and rolling with slipping.

Cylinder rolling down incline
A cylinder rolls down an incline making an angle with the horizontal, without slipping (**Figure 3**). Both the linear speed and the angular velocity (defined as positive in the directions shown) increase with time. To cause such an angular acceleration, the frictional force f must be in the direction shown, in order to provide a torque of the correct sign.

In contrast with the previous example, the friction is opposite to the direction of motion, even though the linear motion is accelerating. Again, you just have to believe the equation of motion: torque is in the same direction as angular acceleration.

1.3 Factors to consider

Thus we have to first deal with two concepts: friction and the condition of no slipping.

2 Friction

2.1 Phenomenological law

Friction between two solid surfaces in contact is very complicated at a microscopic level, and we do not expect a law of friction that is exact and grounded in fundamental principles; all we can expect is a *phenomenological law*, based on observations. However, the equations of motion (such as Newton's second law and its rotational analog) are exact, and are indeed used to infer properties of friction — as we did in the examples in the last Section.

2.2 Limiting friction

At rest
Consider a block resting on a table (**Figure 4**). A horizontal force F is applied to it. The following is observed.

- If $F = 0.1$ N, the block does not move. Thus the friction must be $f = 0.1$ N, in the opposite direction.
- If $F = 0.2$ N, the block still does not move. Thus the friction must be $f = 0.2$ N, in the opposite direction.
- Likewise for $F = 0.3$ N.
- When $F = 0.4$ N, the blocks *just* starts to move. We conclude the *maximum* value of the frictional force in this experimental situation is 0.4 N. .

This simple thought experiment tells us two properties: (a) For the case of the contact being at rest, the law of friction does not tell us the value of f; rather it is to be determined from Newton's second law, in terms of the other forces. (b) The only thing we know from the law of friction is the *limiting friction*, denoted as L: $|f| \leq L$.

In motion
Once there is relative motion between the two surfaces, the frictional force remains at the value L, independent of velocity. (Actually, it drops slightly below L, but we shall ignore this difference here.)

Force versus velocity
Thus the plot of frictional force f versus the velocity v is shown in **Figure 5a**. Friction f is negative (positive) if v is positive (negative).

The vertical part of the graph is the case of contact at rest — the law of friction as stated above does not tell us what the friction is, only its allowed range. The horizontal parts of the graph refer to the case of contact in motion.

In contrast, the viscosity due to a fluid (e.g., a ball moving through air) is much better represented by a different model: the magnitude of the force is proportional to velocity: $f = -bv$ (**Figure 5b**).

Which velocity?
For a block sliding on a table (**Figure 4**), there is only one velocity. For a wheel rolling on a road, different parts of the wheel have different velocities (**Figure 6**): in the normal case the top (bottom) of the wheel has a higher (lower) velocity compared to the center. It is obvious that *only the velocity of the contact point matters*, and the distinction between "at rest" and "in motion" refers to this velocity. We come back to this in the next Section.

2.3 Coefficient of friction

The limiting friction is not an absolute number; it depends on how hard the two surfaces are pressed together — described by the force of *normal reaction* due to the surface, denoted by N. Usually, this is just the weight of the object on top, but is (a) increased if there is some other force pressing on it, and (b) reduced by a factor of $\cos\theta$ if the object is placed on an incline of angle θ, so that only a component of gravity acts normal to the surface. It is not surprising that the limiting friction L increases with normal reaction N, and it is found phenomenologically that they are proportional:

$$L = \mu N \tag{1}$$

where μ is called the *coefficient of friction*.

Table 1 gives illustrative values of the (static) coefficient of friction. The values depend on whether the surfaces are well polished, and can be substantially reduced if the two surfaces are separated by a thin layer of liquid, especially oil.

		μ
aluminum	aluminum	1 to 1.5
brick	wood	0.6
tyre	asphalt	0.72
diamond	diamond	0.1
oak	oak	0.6
wood	wood	0.25 to 0.5

Table 1. Static coefficients of friction. Adapted from
`http://www.engineeringtoolbox.com/friction-coefficients-d_778.html`.
from which more examples can be found.

- We write $L = \mu N$ and not (as in some textbooks) $f = \mu N$, to emphasize that in many cases f can be smaller.
- As mentioned earlier, the frictional force is motion can be a bit smaller. To distinguish between the two cases, we can use a larger *static coefficient* $\mu_{\rm s}$ for the limiting static value L, and a smaller *kinetic coefficient* $\mu_{\rm k}$ for the case of motion. For simplicity, this difference is ignored.

2.4 Examples

A number of examples are presented as Problems. All these involve friction *without* elements of rolling, so that students can gain familiarity with the concepts in a simpler context.

Problem 1
A block of wood of mass 1 kg rests on a table.
(a) A horizontal force of 0.1 N is exerted on the block, and the block does not move. What is the frictional force?
(b) The horizontal force is increased to 0.2 N. The block still does not move. What is the frictional force now?
(c) What can you say about the coefficient of friction? §

Problem 2
A block of wood rests on an incline making an angle θ with the horizontal. The coefficient of friction is μ.
(a) If θ is gradually increased from zero, at what critical angle θ_c would the block start to slide?
(b) If $\theta > \theta_c$, what is the linear acceleration down the incline? Express as a fraction of g. §

Problem 3
The coefficient of friction between the tyres of a car and the road is μ.
(a) What is the maximum possible linear acceleration of the car on a level road, even with the most powerful engine?
(b) If $\mu = 0.72$ (for "standard" tyres on asphalt), what is the corresponding time needed to accelerate from 0 to 60 mph (1 mile = 1.6 km)?
(c) Would this limit apply to cars equipped with rockets? Why (not)? §

Problem 4
A car is travelling very fast and the driver suddenly sees an obstacle ahead. He steps hard on the brakes, and immediately stops the rotation of the wheels; the tyres then skid on the road surface,

leaving a skid mark of 40 m before the car comes to a stop. If the coefficient of friction is $\mu = 0.72$, find the velocity of the car when the brakes were applied. To a good approximation, this is how car speeds are determined at accident sites. §

Problem 5
A uniform ladder rests with its lower end on the ground (coefficient of friction μ) and its upper end against a smooth wall. The ladder makes an angle θ with the vertical. What is the maximum value of θ for the ladder not to slip? §

3 No-slip condition

In this Section we consider the "normal" case of rolling, in which a circular or spherical object (a wheel, a ball) rolls without slipping. Three aspects will be considered: kinematics, force and energy.

3.1 Kinematics

Simple argument
Consider a wheel of radius R, rolling on a road without slipping. Imagine wet paint is put on the rim. Then for every turn of the wheel, i.e., an angular displacement of 2π, a line of paint of length $2\pi R$ will be deposited on the road. In other words, the ratio between linear displacement and angular displacement is R. Dividing by time, the ratio between linear velocity v (of the center of the wheel) and angular velocity ω is also R

$$\boxed{v = R\omega} \tag{2}$$

This is the no-slip condition.

Using relative velocity
A more systematic derivation goes as follows. **Figure 7** shows a wheel whose center C is moving forward with velocity $\vec{v} = v\,\hat{\mathbf{i}}$, and rotating at angular velocity ω. For any point on the rim, the velocity $\vec{u}$ relative to C is $R\omega$ (see the module on *Rotation: Part 1*), but is directed in various directions. The velocity of each point relative to the ground is then $\vec{v} + \vec{u}$.

But for the bottom point B, $\vec{u}$ is in a direction opposite to $\vec{v}$. So the instantaneous velocity of the bottom point, denoted as v_B, is given by

$$v_B \quad = \quad v - R\omega \tag{3}$$

This relationship is true whether or not there is slipping.

The special case of no slipping means $v_B = 0$, which then recovers the condition (2).

3.2 Force

Since the point of contact is at rest, the frictional force f is unknown, except that it cannot be larger than the limiting value: $-L \le f \le L$. It can have any value in that range, determined by other conditions (e.g., Newton's second law).

Example 1
A car has a mass 1 000 kg. What is the frictional force due to the ground acting on the tyres when the car is (a) at rest, (b) accelerating at 1 m s^{-2}, and (c) decelerating at 1 m s^{-2}, assuming no slip in the last two cases?

By applying Newton's second law, the answers are obviously (a) 0 N, (b) $+10^3$ N, (c) -10^3 N. This trivial problem illustrates that, in such cases, the value of f is obtained from Newton's second law, not from the law of friction. §

Beginning students often ask: How can friction be in the forward direction? Should it not always tend to oppose motion? The answer is simple: The road does not "know" about the motion of the car, or of the center of the wheel; it only "knows" about the motion of the point of contact — which is momentarily at rest in all three cases. You can imagine that when the car accelerates, the bottom point is "trying" to move backwards, just as when you run, your shoes are pushing the ground backwards — and friction opposes that backward motion.

3.3 Energy

Go back to Example 1. What is the rate of work done by friction in cases (b) and (c)? The force f is not zero, but the velocity of the point of contact is $v_B = 0$. So the rate of work done is $P = f v_B = 0$. Therefore: in rolling motion without slipping, the frictional force does no work.

In this model, an object that is perfectly circular or spherical would roll forever on a horizontal surface, even if the surface is slightly rough (i.e., $\mu \neq 0$). That is a good approximation: after all, a ball rolls much farther than a block can slide. But this cannot be strictly true: balls eventually come to a stop. Where did we go wrong?

Take the example of car tyres. Because the tyre pressure is finite, the contact cannot be a single point. A better model of a tyre is shown in **Figure 8**, where the angle θ is very small, but non-zero. Obviously the velocity is not strictly zero everywhere on the contact surface; therefore it is not exactly true that no work is done. This explanation is provided just to resolve the paradox. In all problems we encounter in this module, this effect is ignored.

4 Rolling down an incline

4.1 Description of problem

We now deal with one single example in some detail. The situation we consider is shown in **Figure 9a**. A cylinder or sphere of mass M, radius R and moment of inertia $I = \beta M R^2$ is placed on an incline making an angle θ with the horizontal. It is released from a height h, and rolls down the incline without slipping. What is the velocity v of the center of the cylinder or sphere when it reaches the bottom?

The following refers to a cylinder, but by allowing for a parameter β, other cases are included in the same analysis: a solid cylinder ($\beta = 1/2$), a thin cylindrical shell ($\beta = 1$), a solid sphere ($\beta = 2/5$) or a thin spherical shell ($\beta = 2/3$).

It is also convenient to express the answer v in terms of a reference velocity v_0 — the velocity of a hypothetical particle that slides down the same incline without friction. Obviously

$$v_0^2 = 2gh \tag{4}$$

while

$$v = \gamma v_0 \tag{5}$$

where γ is a constant, which being dimensionless, can only depend on β and possibly θ. Actually it will turn out not to depend on θ. Incidentally, this shows that by giving some thought to units and dimensions, we already know a lot about the answer without invoking any laws of dynamics.

4.2 Method 1: energy

The easiest method is to use the conservation of energy. Initially, when the cylinder is at the top of the incline, it has zero KE, and PE given by

$$U = Mgh \tag{6}$$

At the bottom, there are two contributions to the KE: due to the motion of the CM and due to rotation about the CM. In obvious notation:

$$K = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 \tag{7}$$

The formal proof that the KE can be broken up this way was given in an earlier module.

If there is no slip, ω can be expressed in terms of v by (2). Thus both terms in (7) go as v^2, and

$$K = \frac{1}{2}(1+\beta)Mv^2 \tag{8}$$

Since friction does no work when there is no slipping, energy is conserved. Equating (6) and (8) then gives

$$v^2 = \frac{v_0^2}{1+\beta} \tag{9}$$

From this energy perspective, it is easy to see that the angle θ does not matter.

Problem 6
The following four objects are allowed to roll down such an incline in a "race": a solid cylinder, a hollow cylinder, a solid sphere, a thin spherical shell. In what order would they arrive at the bottom? §

4.3 Method 2: forces and torques

The forces acting on the cylinder are shown in **Figure 9b**. The component of gravity $Mg\sin\theta$ acts along the incline; we do not need to deal with the component perpendicular to the incline. A frictional force f acts at the point of contact.

Students often ask: The force f is drawn as pointing up the slope; how do we know that is the direction? We can give three answers.

- The cylinder will roll faster and faster in the direction shown, and that requires a torque due to friction that is pointing up the slope. This is just putting (11) below in words.

- From Method 1, we know that the CM will move slower than a hypothetical point particle not subject to friction, and that means friction must act against gravity. This is just putting (10) in words.
- But the best answer (indeed the real answer) is the following — and this is a general point that is useful in many contexts. By drawing **Figure 9b** this way, we are not making a *factual claim* that f point up the slope; we are only declaring a *choice of convention*: if f points up the slope, its value will be denoted as positive; if f points down the slope, its value will be denoted as negative. Which is actually the case will emerge only at the end, when we find the value of f. It would not matter if we choose "wrongly"; we would only get some negative numbers in the final answer.

Now we can write down the relevant equations. For the linear acceleration a of the CM down the slope,

$$Mg\sin\theta - f = Ma \tag{10}$$

For the angular acceleration α of the rotational motion about the CM, we have

$$\begin{aligned} \tau &= I\alpha \\ fR &= I(\beta MR^2)\alpha \quad (11) \\ f &= \beta M(R\alpha) \quad (12) \end{aligned}$$

Finally, based on the no-slip condition, we have $R\omega = v$, and differentiating this with respect to time gives $R\alpha = a$. Putting this into (12) gives

$$f = \beta Ma \tag{13}$$

When this is put into (10) and this term is moved to the RHS, we get

$$\begin{aligned} Mg\sin\theta &= (1+\beta)Ma \\ a &= \frac{1}{1+\beta}\, g\sin\theta \quad (14) \end{aligned}$$

The velocity after travelling a distance $s = h/\sin\theta$ along the incline is given by

$$\begin{aligned} v^2 &= 2as \\ &= 2\cdot\frac{1}{1+\beta}\, g\sin\theta\cdot\frac{h}{\sin\theta} \\ &= \frac{1}{1+\beta}\cdot 2gh = \frac{v_0^2}{1+\beta} \quad (15) \end{aligned}$$

giving the same answer as before. Note that $\sin\theta$ cancels.

We also need to emphasize one point. The torque equation (11) is written about the CM, which is accelerating. Non-inertial frames are acceptable provided we add a pseudo-force $-ma$, where a is the acceleration of the frame. This is like an additional uniform gravitational force — which importantly acts through the CM. It therefore contributes zero extra torque. This would *not* be true if the rotation is not considered about the CM.

4.4 Method 3: torque about point of contact

Since the point of contact B at the bottom of the cylinder is momentarily at rest, we can regard the whole motion as a rotation about B, without any other translation (**Figure 9c**). To be more specific, take two arbitrary points P, Q on the cylinder. We notice that (a) the lengths of the lines BP, BQ are not changing, (b) the angle PBQ is not changing, and hence (c) the lines BP and BQ must be rotating at the same angular velocity. This is the reason why the motion can be regarded (instantaneously) as a rotation about B.

Also, the point B is instantaneously at rest, so we can consider rotations about this as a fixed axis. Thus

$$\tau' = I'\alpha \tag{16}$$

The torque τ' is caused by the force component $Mg\sin\theta$ acting at a moment arm R from the axis B, thus

$$\tau' = Mg\sin\theta\cdot R \tag{17}$$

On the other hand, the moment of inertia about B can be calculated by the parallel axis theorem as

$$I' = I + MR^2 = (\beta+1)MR^2 \tag{18}$$

Putting (17) and (18) into (16) then yields

$$\begin{aligned} \alpha &= \frac{1}{1+\beta}\cdot\frac{g\sin\theta}{R} \\ a &= R\alpha = \frac{1}{1+\beta}\, g\sin\theta \quad (19) \end{aligned}$$

and the rest of the calculation proceeds as in Method 2.

4.5 Other examples

Problem 7
A pulley is in the shape of a uniform solid cylindrical disk of mass M and radius R. A light string is wound around it, and a mass m is hung at the end of the string (**Figure 10**). Find the linear acceleration a of the mass m, (a) by considering the changes in the PE and the KE after m descends by an amount h, and (b) by considering forces and torques. This problem also involves a combination of translation and rotation, though of different bodies. §

Problem 8
Two wheels of radius R and total mass M, total moment of inertia $I = \beta MR^2$ are connected by an axle of radius R_1 and negligible mass (**Figure 11a**).
(a) A string is wound around the axle and is pulled by a force F; there is of course also a frictional force f (**Figure 11b**). The wheels accelerate without slipping.) Find the acceleration a.
(b) What if there are two such strings and equal forces F pull in opposite directions, but with torques in the same sense (**Figure 11c**)? Pay attention to the signs of f in the two cases.
(c) Where does the increased KE come from? §

5 Examples with slipping

This Section introduces the ideas through just one example.

A bowling ball is set into motion with initial velocity v_0 and initial angular velocity ω_0 (**Figure 12a**). It rolls with slipping on the bowling alley, where the coefficient of friction μ is large enough that the ball ends up rolling *without* further slipping, after a time t_* and with a final velocity v_*. We want to find the time t_* and the velocity v_*.

During the part of the motion before slipping stops, there is a frictional force f acting on the ball (**Figure 12b**). The equations of linear and rotational motion are as follows.

$$\begin{aligned} v(t) &= v_0 - at = v_0 - \frac{f}{M}\,t \\ \omega(t) &= \omega_0 + \alpha t = \omega_0 + \frac{\tau}{I}\,t \\ &= \omega_0 + \frac{fR}{\beta MR^2}\,t \end{aligned} \tag{20}$$

Note that f (if it is positive, i.e., in the direction shown) causes the linear motion to slow down but the rotational motion to speed up. It will be convenient to introduce $u(t) = R\,\omega(t)$ (the velocity of the rim of the ball relative to its center) and also $u_0 = u(0) = R\,\omega_0$.

At the time t_*, the no-slip condition applies: $v(t_*) = R\,\omega(t_*)$, or explicitly

$$\begin{aligned} v_0 - \frac{f}{M}\,t_* &= u_0 + \frac{f}{M\beta}\,t_* \\ v_0 - u_0 &= \frac{f}{M}(1+\beta^{-1})\,t_* \end{aligned} \tag{21}$$

from which we get

$$\begin{aligned} t_* &= \frac{\beta}{1+\beta}\frac{M}{f}(v_0 - u_0) \\ &= \frac{\beta}{1+\beta}\frac{v_0 - u_0}{\mu g} \end{aligned} \tag{22}$$

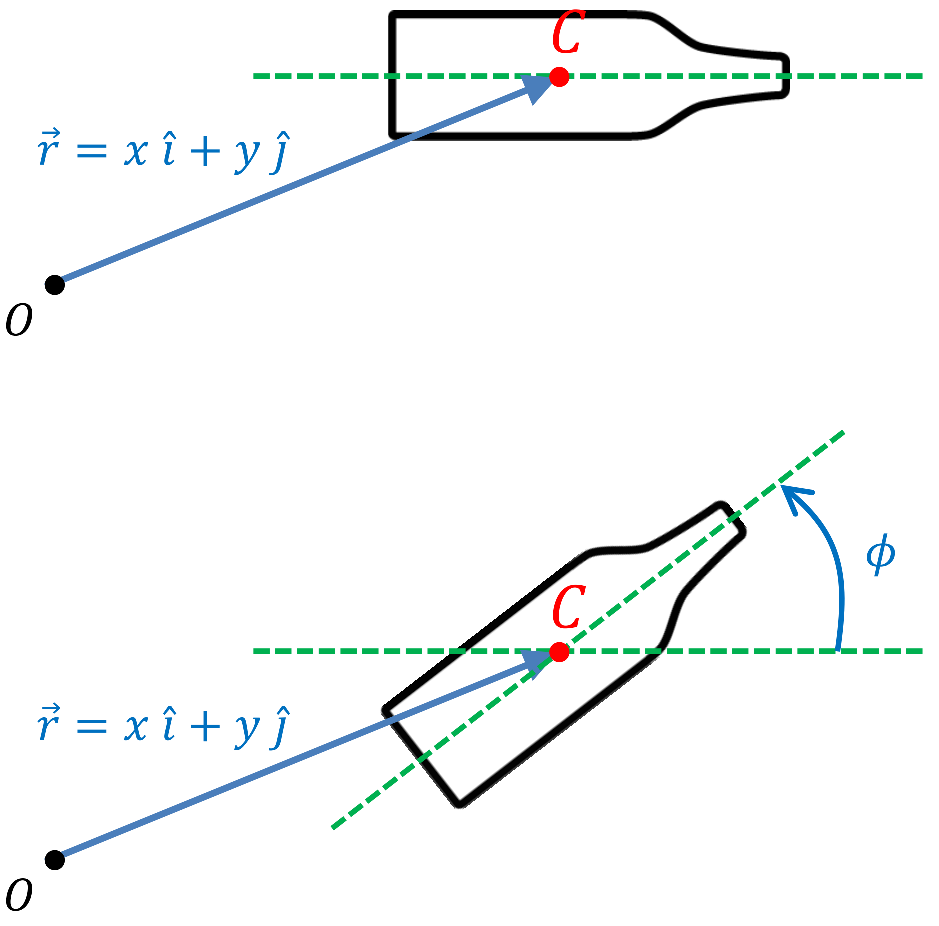
where in the last step we have used $f = \mu Mg$. The first factor is $2/7$ for a solid sphere ($\beta = 2/5$).

The velocity at that time is then

$$\begin{aligned} v_* = v(t_*) &= v_0 - \frac{f}{M}\,t_* \\ &= v_0 - \frac{\beta}{1+\beta}(v_0 - u_0) \\ &= \frac{1\cdot v_0 + \beta\cdot u_0}{1+\beta} \end{aligned} \tag{23}$$

- Slipping is due to the mismatch between $v(t)$ and $u(t)$. Therefore the time taken for slipping to end is $t_* \propto (v_0 - u_0)$.
- The final velocity v_* is a weighted average of v_0 and u_0.

Problem 9
A billiard ball of mass M and radius R, initially at rest, is struck horizontally by a cue stick at a height $h = \gamma R$ above the billiard table. A large force F is exerted over a very short interval Δt.
(a) What must be the value of γ if the ball is to start rolling without slipping? Call this value γ_0 and express it in terms of the parameter β in $I = \beta MR^2$, and also evaluate it for the specific value $\beta = 2/5$ for a solid sphere.
(b) If $\gamma \neq \gamma_0$, find the ratio v/v_0, where v_0 is the initial linear velocity of the ball, and v the eventual velocity after it has come to a state of rolling without slipping. §



C
$\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath}$
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Figure 1

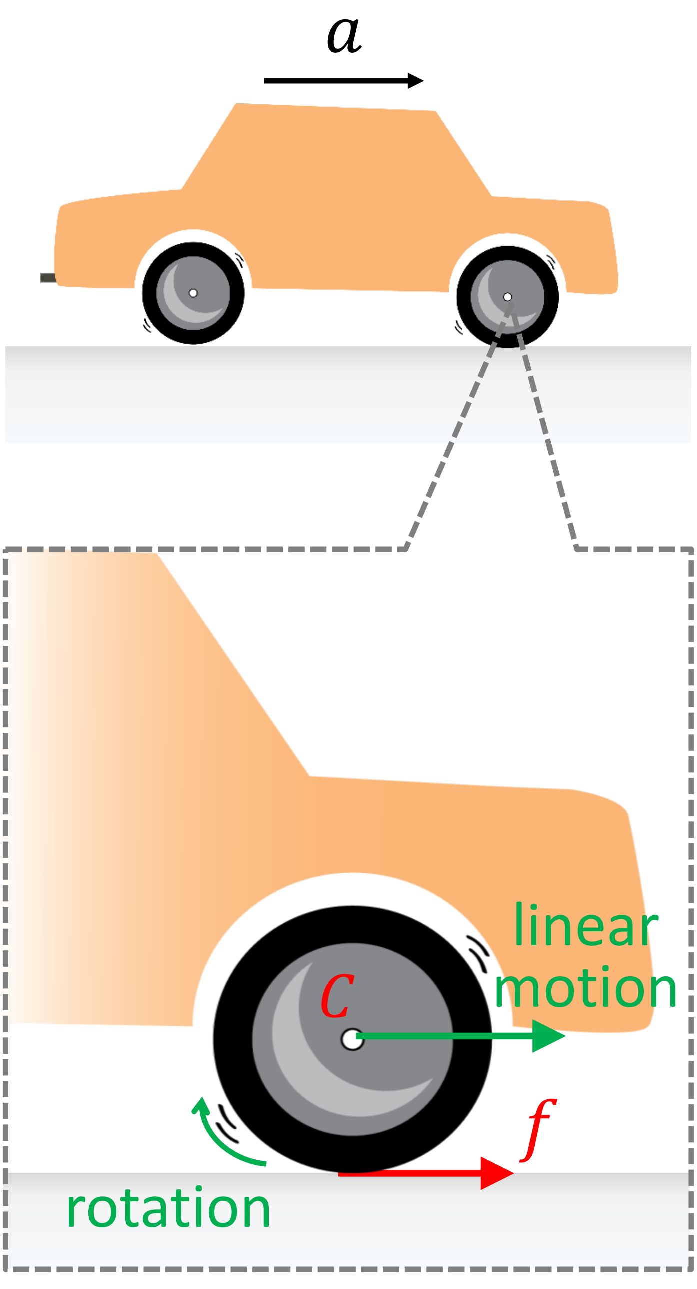


Figure 2

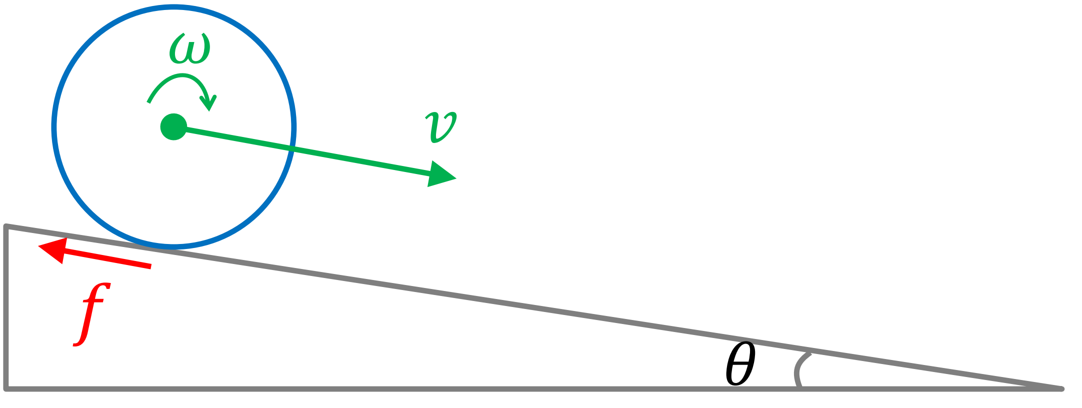
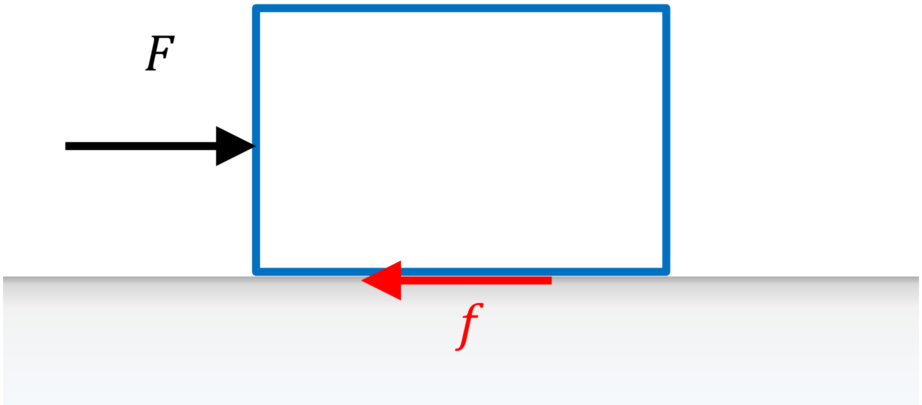


Figure 3



$F = 0.1$ N	at rest	$f = 0.1$ N
$F = 0.2$ N	at rest	$f = 0.2$ N
$F = 0.3$ N	at rest	$f = 0.3$ N
$F = 0.4$ N	begins to move	$f = 0.4$ N is maximum value

Figure 4

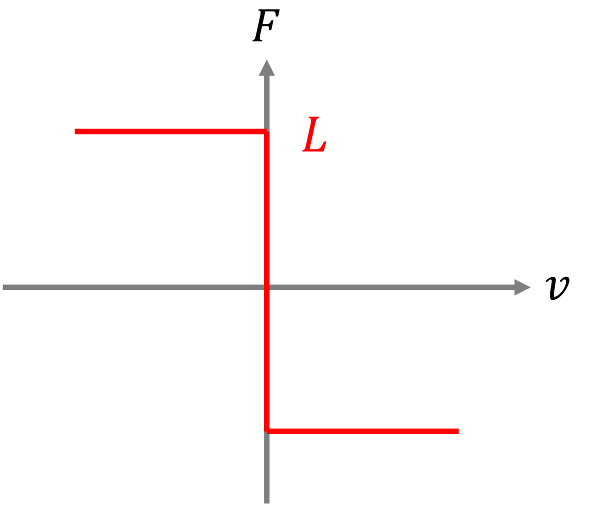


Figure 5a

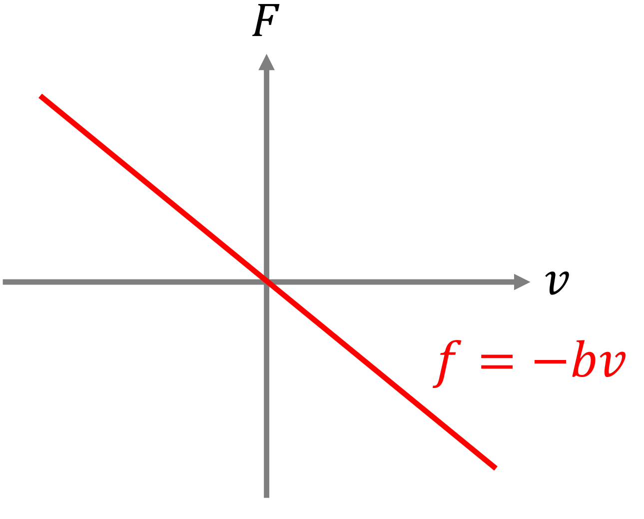


Figure 5b

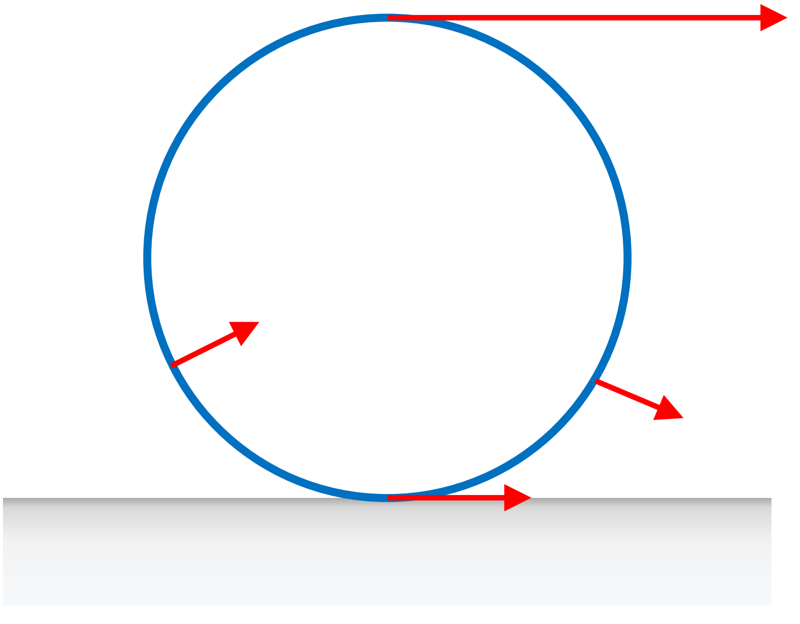
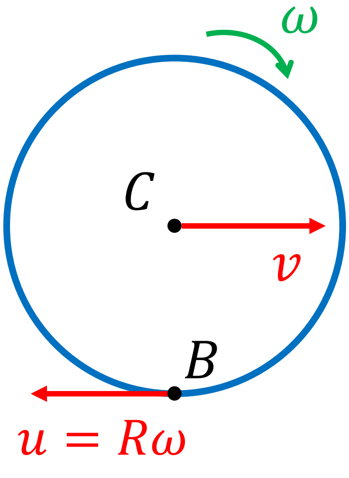


Figure 6



$$v_B = v - u = v - R\omega$$

Figure 7

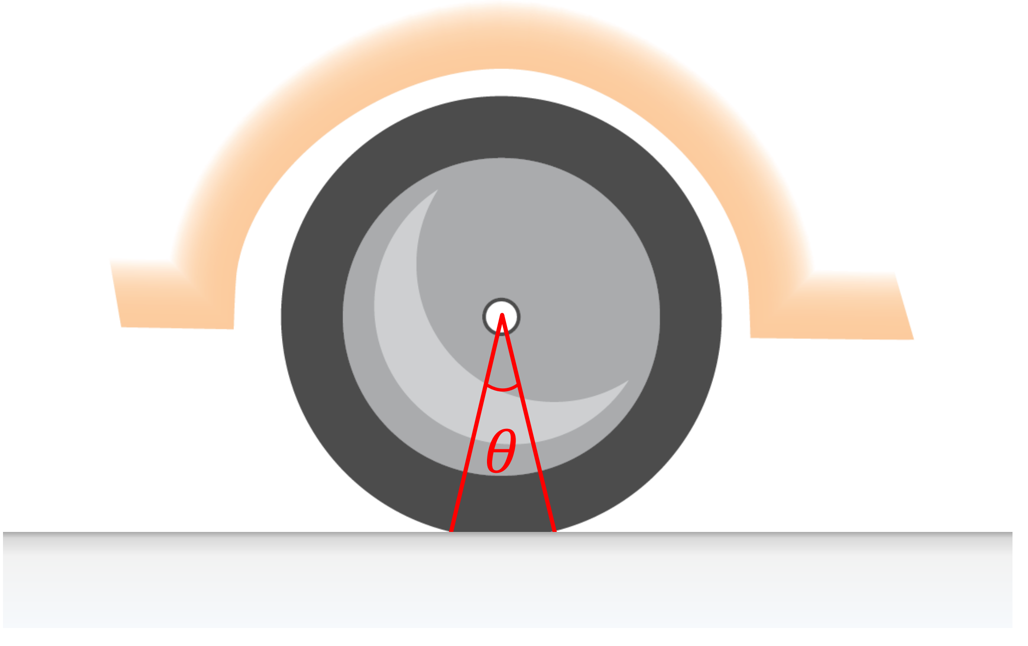


Figure 8

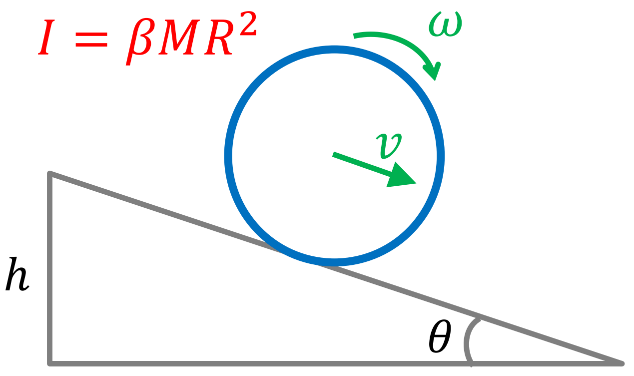


Figure 9a

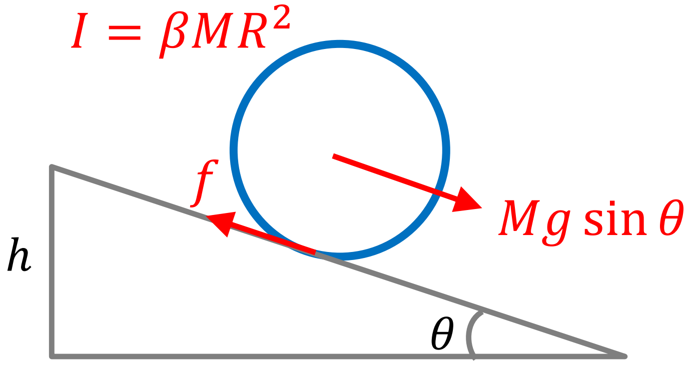


Figure 9b

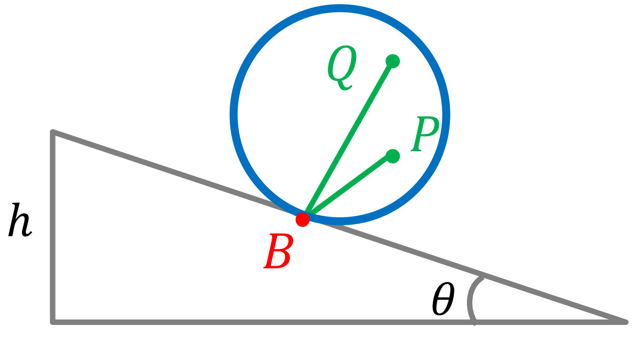


Figure 9c

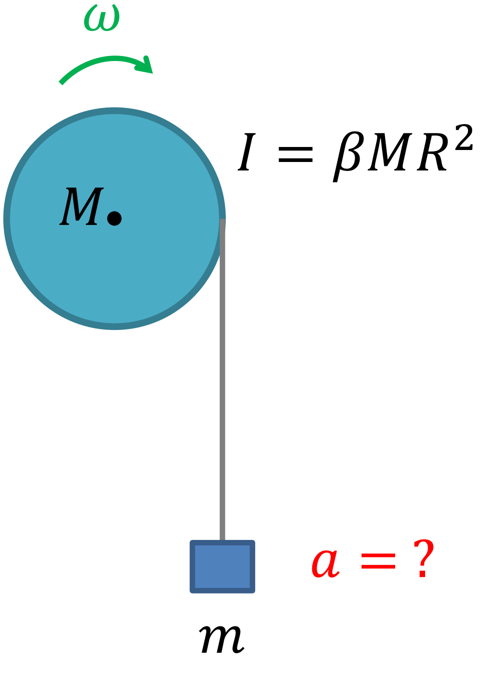


Figure 10

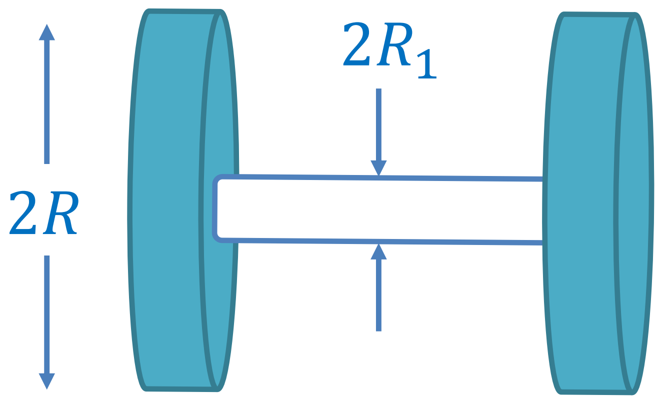


Figure 11a

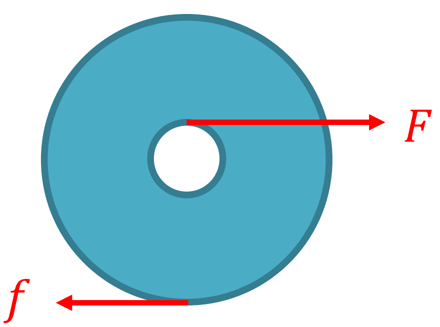


Figure 11b

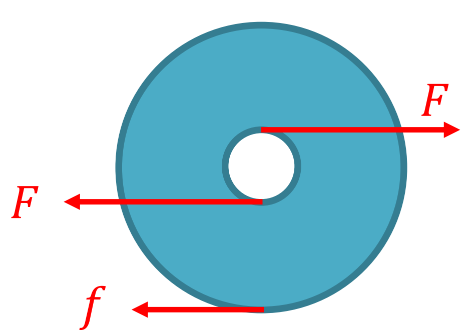


Figure 11c

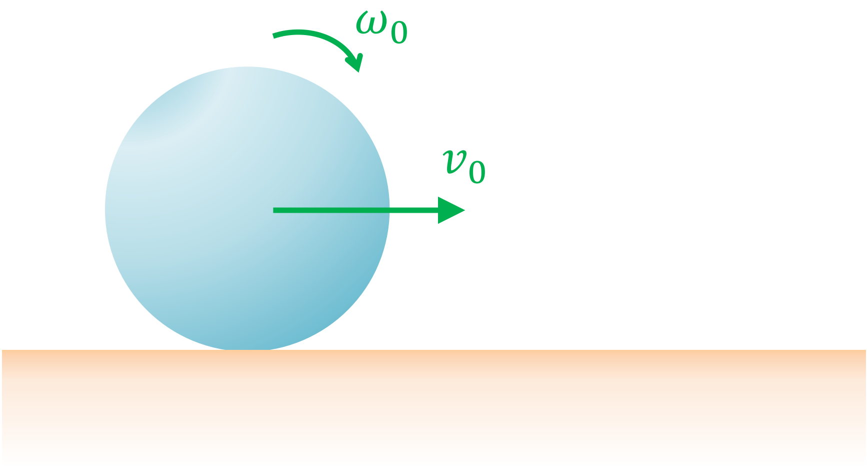


Figure 12a

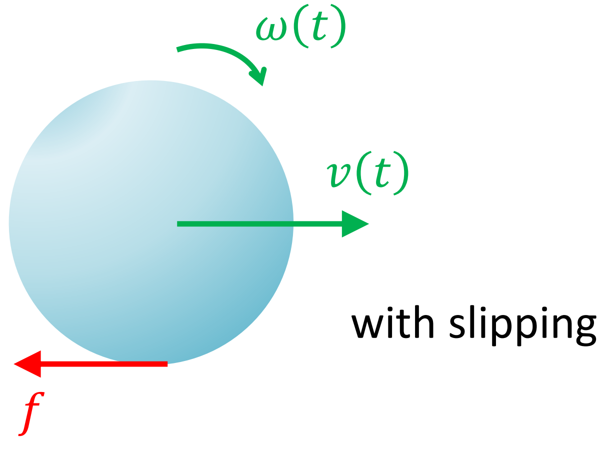


Figure 12b