# Energy: higher dimensions

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*The concepts of potential energy and conservation of energy are formulated in n dimensions. The evaluation of the force from the potential energy leads to the gradient operator.*

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# 1 Potential energy

## 1.1 Definition

This module restricts attention to forces that are *conservative*. If in doubt, check that (see last module)

$$(\mathrm{curl}\,\vec{F})_{ij} = 0 \tag{1}$$

Thus, the work done by the force in going from a reference point $O$ (taken to be fixed and often not indicated explicitly) to a point $\vec{r}$ depends only on $\vec{r}$ and not on the path $\gamma(O, \vec{r})$ linking the two points:

$$\begin{aligned} W(\gamma(O,\vec{r})) &= W(\vec{r}) \\ \int_{\gamma(O,\vec{r})} \vec{F}\cdot d\vec{r} &= \int_O^{\vec{r}} \vec{F}\cdot d\vec{r} \end{aligned} \tag{2}$$

Thus we are able to define the PE as a function of the final position $\vec{r}$:

$$U(\vec{r}) = -\int_O^{\vec{r}} \vec{F}\cdot d\vec{r} \tag{3}$$

Refer to the 1D case for the minus sign: $U$ increases in the direction *opposite* to that of $\vec{F}$. In the case of gravity, $U$ increases upwards, so it can be thought of heuristically as a "height".

## 1.2 Examples

**Example 1**
Suppose the force is an anisotropic harmonic oscillator

$$\vec{F} = -\left(k_1 x\,\mathbf{i} + k_2 y\,\mathbf{j} + k_3 z\,\mathbf{k}\right) \tag{4}$$

First check that it is conservative, e.g.,

$$\begin{aligned} (\mathrm{curl}\,\vec{F})_{xy} &= \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \\ &= \frac{\partial(k_2 y)}{\partial x} - \frac{\partial(k_1 x)}{\partial y} = 0 \end{aligned} \tag{5}$$

Thus we can calculate the PE as follows, using the origin as the reference

$$\begin{aligned} U(x,y,z) &= -\int_0^{(x,y,z)} \vec{F}\cdot d\vec{r} \\ &= \int_0^{(x,y,z)} (k_1 x\,dx + k_2 y\,dy + k_3 z\,dz) \\ &= \frac{1}{2}\left(k_1 x^2 + k_2 y^2 + k_3 z^2\right) \end{aligned} \tag{6}$$

Be careful with the evaluation. If we had encountered a term such as $\int y\,dx$, we would have to ask: "Which path?", because the path determines how $y$ depends on $x$. This does not happen here — a matter of luck in a special case. §

**Example 2**
Find $U(\vec{r})$ corresponding to the central force

$$\vec{F}(\vec{r}) \quad = \quad \frac{k}{r^2}\,\mathbf{e}_r \tag{7}$$

where $\mathbf{e}_r = \vec{r}/r$ is the unit radial vector. This describes a repulsive inverse-square force if $k > 0$ (as between two like charges) or an attractive inverse-square force if $k < 0$ (as between two unlike charges, or gravity between two masses).

Let $\vec{r}$ be described by polar coordinates $(r, \theta, \phi)$. Choose the reference point at infinity. Choose the path along a radius so that $\theta$ and $\phi$ are constant on the path and

$$d\vec{r} \quad = \quad dr\,\mathbf{e}_r \tag{8}$$

You may say: The path goes from infinity to $r$; shouldn't there be a minus sign in (8)? No. The sign will take care of itself, because $dr$ is negative once the limits of integration are specified (lower limit > upper limit). Thus

$$\begin{aligned} & U(\vec{r}) = -\int_{\infty}^{\vec{r}} \vec{F}\cdot d\vec{r} \\ = & \; -\int_{\infty}^{\vec{r}} \vec{F}\cdot\mathbf{e}_r\,dr = -\int_{\infty}^{r} \frac{k}{r^2}\,dr \\ = & \; \left.\frac{k}{r}\right|_{\infty}^{r} = \frac{k}{r} \end{aligned} \tag{9}$$

In the second line we have evaluated $\vec{F}\cdot\mathbf{e}_r$ and this is the only component that matters since the path is chosen to be radial. §

# 2 Force from PE

## 2.1 Force components in terms of partial derivatives

The last Section has given $U(\vec{r})$ as an integral of $\vec{F}$ (see (3)). This Section asks the reverse question: How do we find $\vec{F}$ if $U(\vec{r})$ is given? The formalism here is just the fundamental theorem of vector calculus placed in a particular physical context.

Recall the corresponding derivation in the case of 1D: we compare the PE at two neighbouring points $x$ and $x+\Delta x$. But now we can choose a pair of neighbouring points three ways, being separated in either the $x$, $y$ or $z$ direction.

**Figure 1** illustrates, on the $x$–$y$ plane (i.e., $z$ suppressed), how a point $A = (x, y, z)$ can be compared to $B = (x+\Delta x, y, z)$ or to $C = (x, y+\Delta y, z)$. Taking the first case

$$\begin{aligned} & U(x+\Delta x, y, z) - U(x, y, z) \\ = & \; U(B) - U(A) \\ = & \; -\left(\int_O^B \vec{F}\cdot d\vec{r} - \int_O^A \vec{F}\cdot d\vec{r}\right) \\ = & \; -\int_A^B \vec{F}\cdot d\vec{r} = -\vec{F}\cdot\Delta\vec{r} \end{aligned} \tag{10}$$

In the last step the line integral over a *short* interval is just a simple product, and $\vec{F}$ is evaluated at any point in the interval, say $A$. Since for this comparison, $\Delta\vec{r} = \Delta x\,\mathbf{i}$, the dot product gives

$$\vec{F}\cdot\Delta\vec{r} \quad = \quad F_x\,\Delta x \tag{11}$$

Putting this into (10) and re-arranging terms, we have

$$\begin{aligned} & F_x(x, y, z) \\ = & \; -\frac{U(x+\Delta x, y, z) - U(x, y, z)}{\Delta x} \\ = & \; -\frac{\partial U}{\partial x} \end{aligned} \tag{12}$$

In the last step, upon taking the implied limit of $\Delta x \to 0$, we get a *partial* derivative because $y$ and $z$ are held fixed. Generalizing to the other components, we have

$$\boxed{F_i = -\partial_i U} \tag{13}$$

where $\partial_i = \partial/\partial r_i$ and $r_i$ is any one of the Cartesian components of $\vec{r}$. This is an obvious generalization of the formula for 1D:

$$F(x) \quad = \quad -\frac{dU(x)}{dx} \tag{14}$$

The same argument can be phrased without using integrals:

$$\begin{aligned} & U(B) - U(A) \\ = & \; -[W(O\to B) - W(O\to A)] \\ = & \; -W(A\to B) = -\vec{F}\cdot\Delta\vec{r} \end{aligned} \tag{15}$$

with the subsequent steps the same as before.

**Problem 1**
For every example of $U(\vec{r})$ discussed in the last Section, do the reverse calculation and find the force. §

## 2.2 Gradient operator

Putting the three components together, we have

$$\begin{aligned}\vec{F} &= F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\mathbf{k} \\ &= -\left(\frac{\partial U}{\partial x}\,\mathbf{i} + \frac{\partial U}{\partial y}\,\mathbf{j} + \frac{\partial U}{\partial x}\,\mathbf{k}\right) \\ &= -\left(\mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}\right)U \qquad (16)\end{aligned}$$

In the last line above, we have put the basis vectors in front simply to stress that the differentiation does not affect them.

We are thus led to define the *gradient operator*

$$\text{grad} = \vec{\nabla} \quad \equiv \quad \mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z} \qquad (17)$$

in terms of which

$$\boxed{\vec{F} = -\vec{\nabla}U} \qquad (18)$$

The symbols grad and $\vec{\nabla}$ are used interchangeably.

All these formulas have the obvious generalization to $n$ dimensions, with coordinates $r_i$ and basis vectors $\mathbf{e}_i$, so that

$$\text{grad} = \vec{\nabla} \quad \equiv \quad \mathbf{e}_i\,\frac{\partial}{\partial r_i} \qquad (19)$$

with summation over repeated indices understood.

Incidentally, the gradient is a *vector operator*: it is an operator because it turns one function into another; it is a vector because it has three (or in general $n$) Cartesian components.[1]

**Summary**
The relationship between $\vec{F}$ and $U$ can be summarized as

$$\vec{F} \overset{\text{line int}}{\longrightarrow} -U \quad , \quad U \overset{\text{gradient}}{\longrightarrow} -\vec{F} \qquad (20)$$

This constitutes an example of the fundamental theorem of vector calculus: that the gradient is the inverse of the line integral.

[1] A more proper characterization is that it has the right transformation properties to be a vector.

## 2.3 Curl of gradient is zero

We started with a force field satisfying $(\text{curl}\,F)_{ij} = 0$, and from that constructed $U$. The reverse is also true: If we start with $U$ and construct $\vec{F} = -\vec{\nabla}U$, then it is guaranteed that $(\text{curl}\,F)_{ij} = 0$. In words: *the curl of a gradient is zero.*

**Problem 4**
Prove the above statement. Hint: The mixed partial derivatives in different orders are equal. §

# 3 Equipotential surface

## 3.1 Definition

The concept of equipotential surfaces is best motivated by an example. A mass $m$ is placed in a uniform gravitational field $g$ (say $9.8\,\text{m}\,\text{s}^{-2}$). The $z$-axis points upwards and the $x$–$y$ plane is horizontal. The PE is

$$U(x, y, z) \quad = \quad mgz \qquad (21)$$

All points on the same plane $z = \text{const}$ have the same value of $U$; therefore such a plane is called an *equipotential surface*. **Figure 2** illustrates a family of such surfaces labelled by the values of $U$ ($y$-axis into the page and suppressed).

This example suggests that we can think of $U$ as a "height". Thus, a family of equipotential surfaces are analogous to contour lines of the same height used in showing terrain variation on maps. **Figure 3a** shows such a series of contours for a radial inverse-square repulsive force (centre of map is a peak) while **Figure 3b** shows the situation for a radial inverse-square attractive force (centre is a well). **Figure 4** illustrates the same situation in a 3-dimensional view; but notice that the three dimensions are $(x, y, U)$ and one spatial direction is not drawn.

## 3.2 Properties

We derive three properties.

- The force is perpendicular to the equipotential.
- The magnitude of the force is $|\Delta U|/|\Delta s|$, where $\Delta s$ is the *perpendicular* distance between two nearby potential surfaces with a difference in value $\Delta U$.

- The force points from the surface of higher $U$ to the surface with lower $U$.

These rules allow us to understand the pattern of the force field given a picture of the equipotential surfaces. The three properties are derived below.

**Force is perpendicular to equipotential**
**Figure 5** shows, schematically, an equipotential (dotted line). Because we consider only a small portion of such a surface, the surface can be regraded as flat. Let $A$, $B$ be two neigbouring points on the surface with separation $\Delta\vec{r}$, and let $\vec{F}$ be the force, for the moment assumed to be at an arbitrary direction. The difference in PE between the two points is

$$\begin{aligned} 0 = \Delta U = U(B) - U(A) \\ = \quad \vec{F} \cdot \Delta\vec{r} \end{aligned} \tag{22}$$

Thus we conclude $\vec{F}$ and $\Delta\vec{r}$ are perpendicular; and this holds for any $\Delta\vec{r}$ lying along the equipotential surface. Thus, the force is perpendicular to the equipotential surface.

**Force is change of $U$ per perpendicular distance**
**Figure 6** shows two nearby equipotential surfaces, with PE values $U$ and $U + \Delta U$. The point $A$ lies on one surface, and $C$ on the other, with $\overline{AC}$ perpendicular to the surfaces, and having a length $\Delta s$. Now

$$\Delta U = U_C - U_A = -\vec{F} \cdot \Delta\vec{r} \tag{23}$$

But $\vec{F}$ is along the same direction as $\Delta\vec{r} = \overline{AC}$, so

$$|\vec{F} \cdot \Delta\vec{r}| = F|\Delta\vec{r}| = F\,\Delta s \tag{24}$$

thus giving for the magnitude

$$F = \frac{|\Delta U|}{|\Delta s|} \tag{25}$$

The third property is obvious.

# 4 Conservation of energy

The laws in this Section can be separated into two levels:

- The relationship between work and kinetic energy (KE), valid for *all* forces.
- The conservation of energy relating KE and PE, applicable *only* to conservative forces.

In both cases the derivation is a simple generalization of the 1D case.

## 4.1 Work and KE

Here we prove that *the work done by the net force acting on an object is equal to the increase in the KE.*

**Constant force**
First consider a constant net force $\vec{F}$. A mass $m$ moves a distance $\Delta\vec{r}$ in a time $\Delta t$, with initial velocity $\vec{v}_1$ and final velocity $\vec{v}_2$. Using the average velocity $\vec{V}$

$$\Delta\vec{r} = \vec{V}\,\Delta t = \frac{1}{2}(\vec{v}_2 + \vec{v}_1)\,\Delta t \tag{26}$$

while Newton's second law gives

$$\vec{F} = m\vec{a} = m\,\frac{\vec{v}_2 - \vec{v}_1}{\Delta t} \tag{27}$$

Taking the dot product of (26) and (27) eliminates $\Delta t$:

$$\begin{aligned} \vec{F} \cdot \Delta\vec{r} &= \frac{1}{2}m(\vec{v}_2 + \vec{v}_1) \cdot (\vec{v}_2 - \vec{v}_1) \\ &= \left(\frac{1}{2}mv_2^2\right) - \left(\frac{1}{2}mv_1^2\right) \end{aligned} \tag{28}$$

Note $v_i^2 = \vec{v}_i \cdot \vec{v}_i$. Define kinetic energy (KE) $K$ in the usual way

$$K = \frac{1}{2}mv^2 \tag{29}$$

and we get

$$W(1 \to 2) = K_2 - K_1 \tag{30}$$

The above derivation has used the formulas for constant acceleration, and is therefore valid (only) for a constant force.

**General force**
For a general (non-constant) force, chop the motion into short segments, in each of which the force can be regarded as constant. Let us illustrate with two segments:

$$\begin{aligned} W(1 \to 3) &= W(1 \to 2) + W(2 \to 3) \\ &= (K_2 - K_1) + (K_3 - K_2) \\ &= K_3 - K_1 \end{aligned} \tag{31}$$

In the above, the first equal sign relies on the additive property of $W$; the second equal sign relies on (30) for a *short* segment in which the force is constant. With the last equal sign, all reference to the intermediate situation cancels, and the RHS depends only on the initial and final states of motion. It is obvious how this argument generalizes to more segments.

The following derivation may look simpler, but is exactly the same idea expressed in another language.

$$\begin{aligned} W(i \to f) = \int_i^f \vec{F} \cdot d\vec{r} = \int_i^f \frac{d}{dt}(m\vec{v}) \cdot (\vec{v}\,dt) \\ = \int_i^f \frac{d}{dt}\left(\frac{1}{2}mv^2\right) \cdot dt \\ = \left.\frac{1}{2}mv^2\right|_i^f = K_f - K_i \end{aligned} \tag{32}$$

In the above, the motion is between an initial time $i$ and a final time $f$. We have used the identity

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = \frac{m}{2}\frac{d}{dt}(\vec{v}\cdot\vec{v}) \\ = m\vec{v} \cdot \frac{d\vec{v}}{dt} \end{aligned} \tag{33}$$

In doing the integral, we have cancelled $dt$ in

$$\frac{d}{dt}(\ldots)\,dt = (\ldots) \tag{34}$$

which can be regarded as the fundamental theorem of calculus (integration and differentiation are inverse operations); it is also the analog of cancelling $\Delta t$ in deriving (28).

## 4.2 Work and PE

First suppose there is only one force $\vec{F}$, which is conservative and associated with potential energy $U$. Then

$$\begin{aligned} W(i \to f) = \int_i^f \vec{F} \cdot d\vec{r} \\ = \int_O^f \vec{F} \cdot d\vec{r} - \int_O^i \vec{F} \cdot d\vec{r} = -(U_f - U_i) \end{aligned} \tag{35}$$

The minus sign comes from the definition of $U$. Combining (32) with (35) then gives

$$\begin{aligned} K_f - K_i &= -(U_f - U_i) \\ K_i + U_i &= K_f + U_f \end{aligned} \tag{36}$$

In other words the total energy

$$E = K + U \tag{37}$$

is the same at the initial and final times: energy is conserved.

If there are several conservative forces, the above still holds for $F$ being the net force and $U$ being the total PE, which can be expressed as the sum of PEs due to each force — here relying on the fact that $W$ is additive.

Thus we get the second statement: *If the forces are conservative, then total energy $E$ is conserved.*

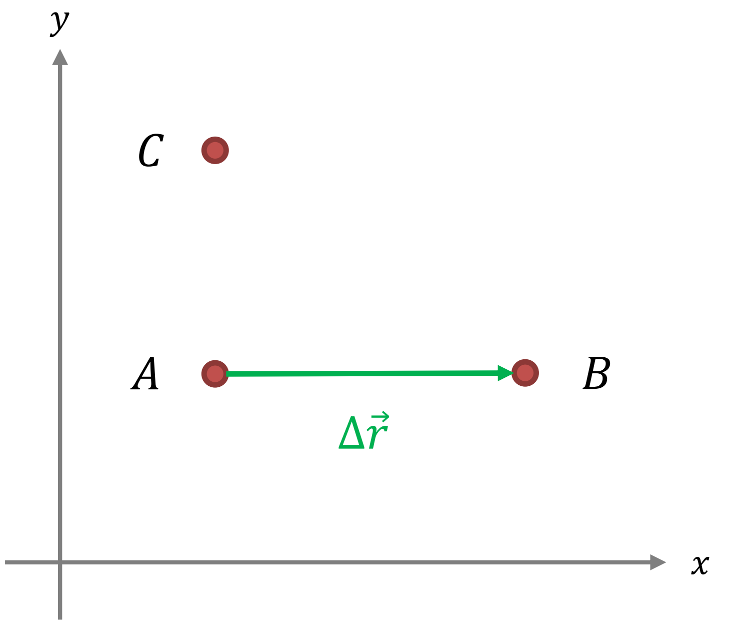


Figure 1

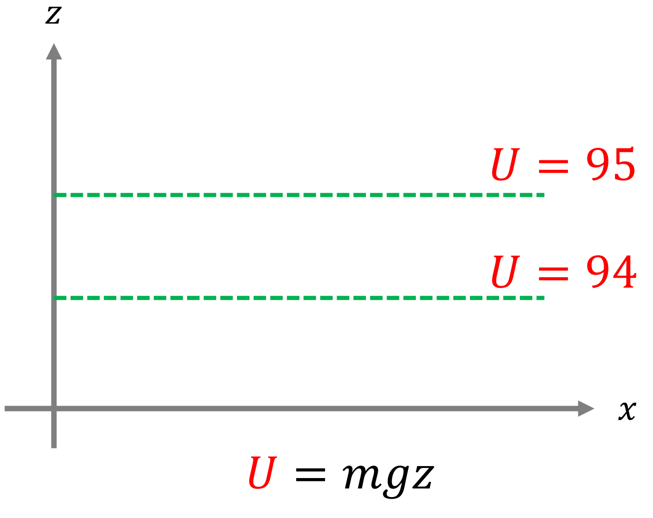


Figure 2

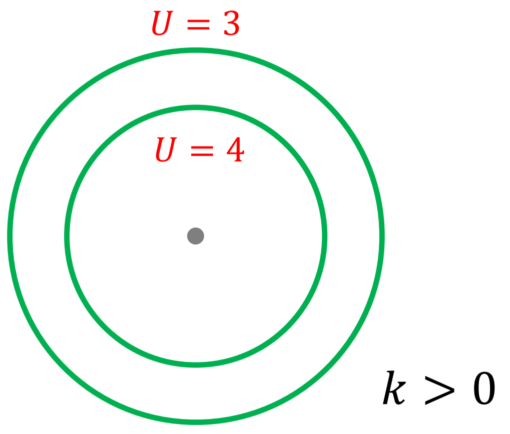


Figure 3a

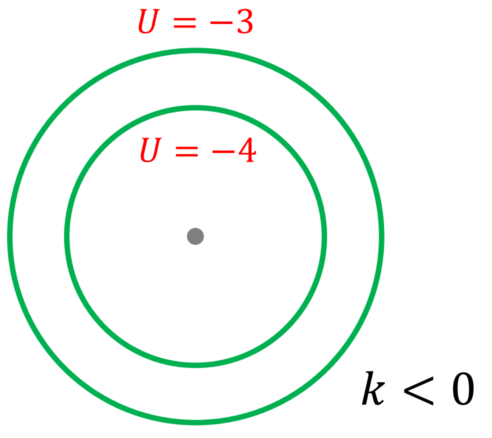


Figure 3b

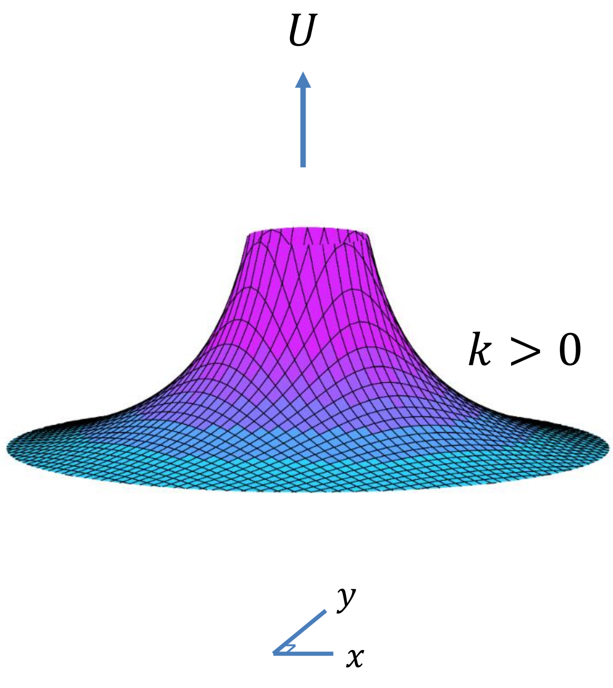


Figure 4a

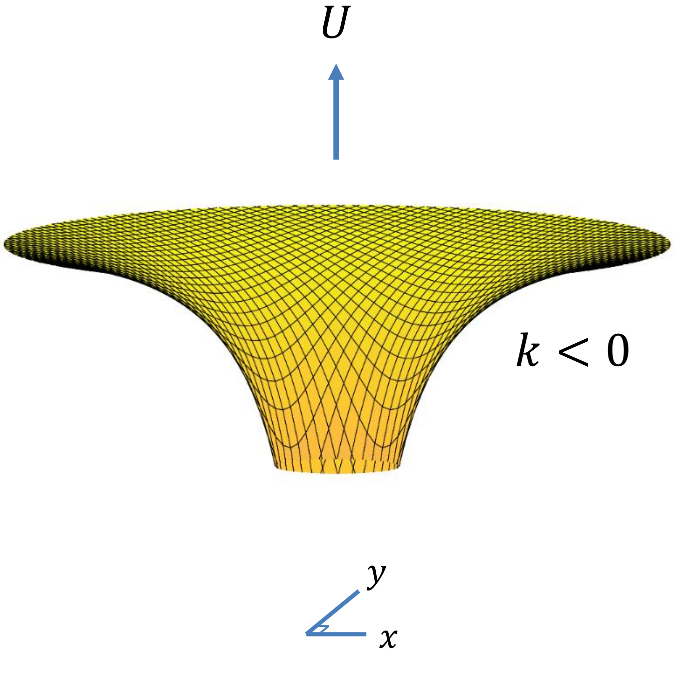


Figure 4b

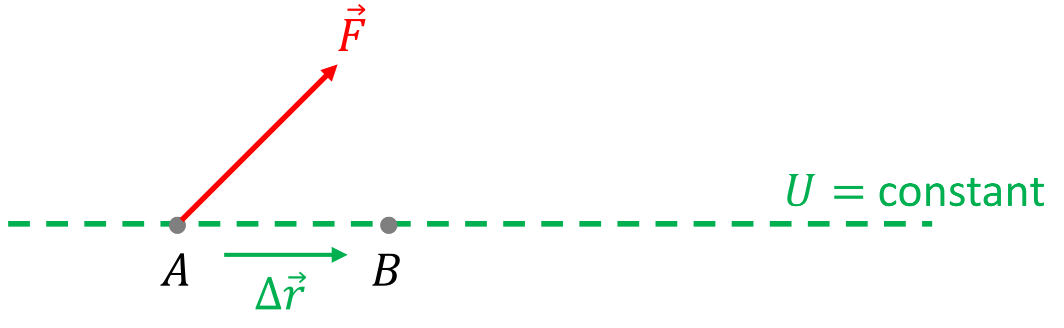


Figure 5

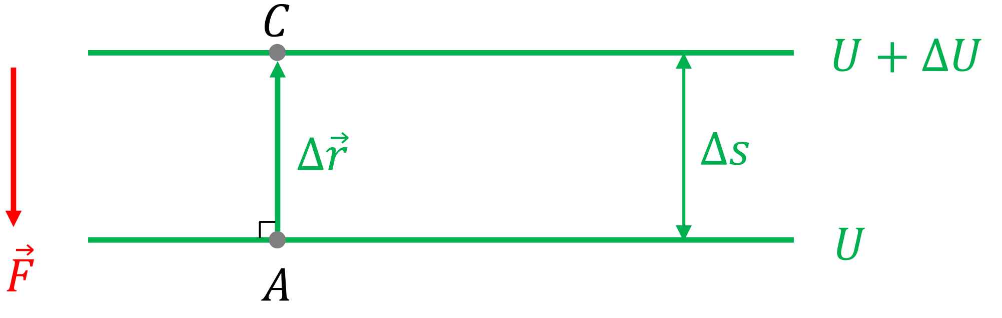


Figure 6