Complex variables: Part 2

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This module introduces some advanced topics. These are not needed in elementary physics. Only a sketch is provided, to convey a taste of the power of complex analysis. The full apparatus will take more effort to learn.

Contents

1 Differentiation

1.1 Rules

The derivative is the limit of a ratio:

$$\frac{df}{dz} = \lim_{\Delta z\to 0}\frac{\Delta f}{\Delta z} \tag{1}$$

This expression involves two subtractions and one division. But since complex numbers obey the same arithmetic rules as real numbers, differentiation proceeds in exactly the same way.[1]

For example:

$$\begin{aligned}\frac{d}{dz}z^n &= nz^{n-1}\\ \frac{d}{dz}e^{iz} &= ie^{iz}\\ \frac{d}{dz}\frac{1}{z} &= -\frac{1}{z^2}\\ \frac{d}{dz}\frac{e^{iz}-1}{z} &= \frac{ie^{iz}}{z}-\frac{e^{iz}-1}{z^2}\\ &= \frac{e^{iz}(iz-1)+1}{z^2}\end{aligned} \tag{2}$$

The derivative of $1/z$ exists except at $z = 0$. In the last example, the point $z = 0$ is perfectly fine. A complex function is said to be *analytic* if it is differentiable (at a point or in a region).

Obviously the sum, difference, product and quotient of analytic functions are also analytic, in the case of the quotient excluding points where the denominator vanishes. A power series is the limiting case of a sum, and every term $a_n z^n$ is differentiable. Thus a function represented by a power series is analytic at all points where the series converges.[2]

1.2 Cauchy–Riemann conditions

Two real functions of two real variables
The above looks very simple, but analyticity imposes strong conditions which are best revealed when we consider a complex function $f(z)$ as two real functions of two real variables: write z as $x+yi$ and also

$$f = u(x,y)+v(x,y)i \tag{3}$$

[1] There is an implicit condition: the function is smooth enough so that the limit exists. In almost all cases in physics, the smoothness condition can be taken for granted. In fact, we shall often adopt a sloppy notation and not write the limit sign, with the understanding that Δ denotes differences that are infinitesimal.

[2] Strictly speaking, only if the series converges uniformly, but we shall not worry about this nicety here.

Example 1
Decompose $f(z) = z^2$ into two real functions.

$$\begin{aligned} f(z) &= (x+yi)^2 \\ &= (x^2-y^2)+2xyi \\ u(x,y) &= x^2-y^2 \\ v(x,y) &= 2xy \end{aligned} \tag{4}$$

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Example 2
Decompose $f(z) = e^{iz}$ into two real functions.

$$\begin{aligned} f(z) &= e^{iz} = e^{ix}\,e^{-y} \\ &= (\cos x + i\sin x)\,e^{-y} \\ u(x,y) &= e^{-y}\cos x \\ v(x,y) &= e^{-y}\sin x \end{aligned} \tag{5}$$

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Partial derivatives: a reminder
For a function of several variables (such as $u(x,y)$), the *partial derivative* with respect to one variable is just like an ordinary derivative, but treating all the other variables as constants. Partial derivatives are denoted by the symbol ∂ instead of d. Thus

$$\frac{\partial u}{\partial x} = \lim_{\Delta x\to 0}\frac{u(x+\Delta x,y)-u(x,y)}{\Delta x} \tag{6}$$

and likewise for $\partial/\partial y$. In the following we shall often adopt a sloppy notation and not write the limit.

Mixed partial derivative
Mixed partial derivatives in different orders are equal[3]

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y} = \frac{\partial}{\partial y}\frac{\partial}{\partial x} \tag{7}$$

The proof is straightforward if reduced to differences. Let a,b denote small quantities that go to zero.

$$\begin{aligned} &\frac{\partial u(x,y)}{\partial x} \\ =\ & (2a)^{-1}[u(x+a,y)-u(x-a,y)] \\ & \frac{\partial^2 u(x,y)}{\partial y\partial x} = \frac{\partial}{\partial y}\frac{\partial u(x,y)}{\partial x} \end{aligned}$$

$$\begin{aligned} &= (2b)^{-1}\left\{\frac{\partial u(x,y+b)}{\partial x} - \frac{\partial u(x,y-b)}{\partial x}\right\} \\ &= (4ab)^{-1}\{[u(x+a,y+b)-u(x-a,y+b)] \\ &\quad -[u(x+a,y-b)-u(x-a,y-b)]\} \\ &= (4ab)^{-1}(u_{++}+u_{--}-u_{+-}-u_{-+}) \end{aligned} \tag{8}$$

in obvious notation. The mixed derivative in the other order gives the same result.

Two ways of taking the difference
A derivative compares function values at two nearby points. But on the complex plane, there are two ways to take this difference, with the nearby points separated either horizontally ($\Delta z = \Delta x$) or vertically ($\Delta z = i\Delta y$). In the first case

$$\begin{aligned} \frac{df}{dz} &= \frac{u(x+\Delta x,y)-u(x,y)}{\Delta x} \\ &\quad + \frac{iv(x+\Delta x,y)-iv(x,y)}{\Delta x} \\ &= \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x} \end{aligned} \tag{9}$$

In the second case

$$\begin{aligned} \frac{df}{dz} &= \frac{u(x,y+\Delta y)-u(x,y)}{i\Delta y} \\ &\quad + \frac{iv(x,y+\Delta y)-iv(x,y)}{i\Delta y} \\ &= -i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y} \end{aligned} \tag{10}$$

But (9) and (10) must be equal, giving the *Cauchy–Riemann conditions*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{11}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{12}$$

Problem 1
Check that u and v in Examples 1 and 2 satisfy the Cauchy–Riemann conditions. §

Real and imaginary parts are mutually dependent
The Cauchy–Riemann conditions mean that u determines v up to a constant, and vice versa.

Example 3
Suppose we know

$$u(x,y) = e^{-y}\cos x \tag{13}$$

[3]Subject to smoothness conditions which we shall not worry about.

From (11)

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = -e^{-y}\sin x \tag{14}$$

Next integrate this with respect to y; remember that in this process x is regarded as a constant. Thus

$$v = e^{-y}\sin x + C(x) \tag{15}$$

There is a "constant" of integration — constant as far as y is concerned, so it can be a function of x. To determine $C(x)$, differentiate with respect to x:

$$\frac{\partial v}{\partial x} = e^{-y}\cos x + C'(x) \tag{16}$$

Use (12) for the LHS:

$$-\frac{\partial u}{\partial y} = e^{-y}\cos x + C'(x) \tag{17}$$

Since u is known, the LHS can be evaluated and it cancels the first term on the RHS, giving $C' = 0$. Thus

$$v = e^{-y}\cos x + C \tag{18}$$

where C is now truly a constant, i.e., independent of both x and y. We recover the function v in Example 2, up to an additive constant. §

Problem 2
(a) Let $u = x^2 - y^2$, Determine v.
(b) Repeat for $u = x^3 - 3xy^2$. §

But wait a minute. Were we lucky? Look at (17) and the analogous equation encountered in Problem 2. What if the LHS and the first term on the RHS fail to cancel completely, leaving something depending on y? Then it cannot be equal to $C'(x)$. As an example, see what happens if you let $u = e^{-y}\cos 2x$. We come back to this issue in the next subsection.

1.3 Harmonic functions

Analytic function leads to harmonic functions
The Cauchy–Riemann conditions also imply conditions on each of u and v.

$$\begin{aligned} \frac{\partial}{\partial x}(11) &\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x\,\partial y} \\ \frac{\partial}{\partial y}(12) &\Rightarrow \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y\,\partial x} \end{aligned} \tag{19}$$

Add and note that the mixed partial derivatives on the RHS are the same. Thus

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u(x,y) = 0 \tag{20}$$

The same holds for v. Functions satisfying this condition (in 2D) are said to be *harmonic*.

Thus the real part of an analytic function is heavily constrained; the same is true of the imaginary part. This constraint is just the Laplace equation in 2D, satisfied by an electrostatic potential in vacuum.

Example 4
Express the harmonic function $u = \Re\, z^n$ in terms of polar coordinates.

Thinking of (x, y) as a point in 2D plane and use the notation $r = |z| = \sqrt{x^2 + y^2}$. Thus

$$\begin{aligned} z &= re^{i\theta} \\ z^n &= r^n e^{in\theta} \\ u &= \Re\, z^n = r^n \cos n\theta \end{aligned} \tag{21}$$

Thus: if a harmonic function depends on the radius as r^n, then it must depend on the angle as $\cos n\theta$ (or $\sin n\theta$ if we consider $-iz^n$). §

Problem 3
Construct another harmonic function from $u = \Re\,\sin z/z$. Hint: Write the sin in terms of exponentials, and then $z = x + yi$. §

Harmonic function leads to analytic function
We now prove that if $u(x, y)$ is a harmonic function, then it can be written as the real part of an analytic function $f(z)$. All we need to do is construct v as in Example 3, but in general terms.

Start with $u(x, y)$ and use

$$\begin{aligned} \frac{\partial v(x,y)}{\partial y} &= \frac{\partial u(x,y)}{\partial x} \\ v(x,y) &= \int_0^y \frac{\partial u(x,y')}{\partial x}\, dy' + C(x) \\ \frac{\partial v(x,y)}{\partial x} &= \int_0^y \frac{\partial^2 u(x,y')}{\partial x^2}\, dy' + C'(x) \\ C'(x) &= -\frac{\partial u(x,y)}{\partial y} \\ &\quad - \int_0^y \frac{\partial^2 u(x,y')}{\partial x^2}\, dy' \end{aligned} \tag{22}$$

where in the last step we have used (12). The last equation is consistent (and can be integrated to give

$C(x)$) if and only if the RHS is independent of y; so we check

$$\frac{\partial\,(\mathrm{RHS})}{\partial y} \stackrel{?}{=} 0 \tag{23}$$

This is precisely the harmonic condition on u.

2 Integration

2.1 Rules

An integral $\int f(z)\,dz$ is just the limiting value of $\sum f(z)\,\Delta z$ — involving multiplication and addition. Since complex numbers obey the same rules of arithmetic as real numbers, exactly the same rules of integration apply. For example,

$$\begin{aligned} \int z^n\,dz &= \frac{1}{n+1} z^{n+1} \\ \int e^{az}\,dz &= a^{-1} e^{az} \\ \int \cos az &= a^{-1} \sin az \end{aligned} \tag{24}$$

There is something deep lurking behind simple formulas like these. To appreciate the subtlety, first consider the case of a definite integral of a real variable:

$$F(\zeta) = \int_0^\zeta f(x)\,dx \tag{25}$$

We have arbitrarily taken the lower limit to be 0; the same argument below applies for any fixed lower limit. There is only one way to go from 0 to ζ along the real line.[4] So there is no need to specify the path for the integral. But now take the analogous case of complex variables

$$F(\zeta) = \int_0^\zeta f(z)\,dz \tag{26}$$

where ζ is some complex number. There are many different paths γ linking 0 to ζ; **Figure 1a** shows two of them, γ_1 and γ_2. The innocuous formulas such as (24) imply that the answer depends only on the endpoint ζ, and not on the path γ. This in turn means that the integral along

$$\gamma = \gamma_1 - \gamma_2 \tag{27}$$

is zero, where the notation means the path along γ_1 and then in the reverse direction along γ_2; see **Figure 1b**. Such a curve γ is *closed*, and we indicate an integral along a closed curve by a small circle on the integral sign. Thus we expect

$$\oint f(z)\,dz = 0 \tag{28}$$

provided f is analytic everywhere inside the closed curve. We now prove this statement explicitly. Although it does not matter in the above formula, the convention for closed-loop integrals is always to go along the path counter-clockwise.

2.2 Cauchy integral theorem

We now prove (28), called the *Cauchy integral theorem*.

A small rectangle
First consider an infinitesimal rectangle centered at z_0, with width $2a$ and height $2b$. Thus the four corners are $z = z_0 \pm a \pm bi$. We enumerate the midpoints z_m and the lengths Δz of the four sides of the rectangle.

side	z_m	Δz
bottom	$z_0 - bi$	$2a$
right	$z_0 + a$	$2bi$
top	$z_0 + bi$	$-2a$
left	$z_0 - a$	$-2bi$

Table 1. The midpoints and the lengths of the four sides of the rectangle.

$$\begin{aligned} &\oint f(z)\,dz = \sum f(z_m)\Delta z \\ =\ & [f(z_0 - bi) - f(z_0 + bi)](2a) \\ & + [f(z_0 + a) - f(z_0 - a)](2bi) \\ =\ & [(df/dz)(-2bi)](2a) \\ & + [(df/dz)(2a)](2bi) = 0 \end{aligned} \tag{29}$$

The point is that whether one takes differences horizontally or vertically, it involves the same df/dz.

[4] Of course one can go to some $\zeta' > \zeta$ and then double back, but the part ζ to ζ' is traversed twice, in opposite directions, and contributes zero to the integral.

Any closed curve
It is easy to generalize to other closed curves by combining many small rectangles, as illustrated in **Figure 2**. The "internal" sides cancel, and only the integral over the boundary remains. Thus we prove the Cauchy integral theorem.

In one sense we can say that the proof is unnecessary: formulas such as (24) show explicitly that the value of the integral depends only on the endpoint, not on the path.

2.3 Residue and meromorphic functions

What if the function is not analytic?

First-order pole
Consider the function

$$f(z) = \frac{1}{z-\zeta} \tag{30}$$

which has a first-order pole at ζ. Let γ be any closed curve that goes round ζ once in the counter-clockwise direction (**Figure 3**), and consider the closed loop integral

$$\oint_\gamma f(z)\,dz \tag{31}$$

First, the curve can be replaced by γ' (**Figure 3**), which is a small circle of radius r around ζ: the difference between the two is γ'' (**Figure 4**), inside which the function is analytic.[5] On the circle, let

$$\begin{aligned} z-\zeta &= re^{i\theta} \\ dz &= ire^{i\theta}\,d\theta \\ \oint \frac{dz}{z-\zeta} &= \int_0^{2\pi} \frac{ire^{i\theta}\,d\theta}{re^{i\theta}} \\ &= \int_0^{2\pi} i\,d\theta = 2\pi i \end{aligned} \tag{32}$$

Higher-order pole
Consider the function

$$f(z) = \frac{1}{(z-\zeta)^n} \tag{33}$$

with $n > 1$. By the same argument we get

$$\begin{aligned} \oint \frac{dz}{(z-\zeta)^n} &= \int_0^{2\pi} \frac{ire^{i\theta}\,d\theta}{r^n e^{in\theta}} \\ &= \int_0^{2\pi} ir^{1-n}\,e^{i(1-n)\theta}\,d\theta \\ &= ir^{1-n}\frac{1}{i(1-n)}\left. e^{i(1-n)\theta}\right|_0^{2\pi} = 0 \end{aligned} \tag{34}$$

So for poles of various orders, the case $n = 1$ is very special.

Meromorphic functions
Suppose a function has a pole of some finite order, and can be expanded as

$$\begin{aligned} f(z) &= \frac{b_N}{(z-\zeta)^N} + \ldots + \frac{b_1}{z-\zeta} + P(z) \\ P(z) &= \sum_{n=0}^{\infty} a_n (z-\zeta)^n \end{aligned} \tag{35}$$

In other words, apart from the inverse powers shown, the rest is a power series $P(z)$ which is analytic. Such a series is called a *Laureant series*, and captures a large class of functions. A function that is (a) analytic except at a finite set of points ζ_j and (b) has a Laureant series expansion at each such point is said to be *meromorphic.*

Integral around a pole
Take the function $f(z)$ in (35) and consider $\oint_\gamma f(z)\,dz$ around a closed curve γ going around the pole ζ once in the counter-clockwise sense. Integrate term by term. Of the inverse powers, only the b_1 term survives; the power series is analytic and gives zero for a closed curve. Hence

$$\oint f(z)\,dz = 2\pi i\,b_1 \tag{36}$$

This is more often written as

$$\frac{1}{2\pi i}\oint f(z)\,dz = \mathrm{Res}\,(\zeta) \tag{37}$$

where the *residue*, denoted by Res, is simply the coefficient b_1 in the Laureant series.[6] If the path encloses several poles, then we add up all the residues.

[5] Note that the pole at ζ lies *outside* the curve. In other words, if the curve γ'' is shrunk to a point, it does not cross the pole.

[6] Of course the residue and the Laureant series refer to a specific function $f(z)$ which is understood. If there is any ambiguity, then the notation should distinguish which function we are talking about.

Example 5
Find

$$I = \oint \frac{1+2z^2}{\sin \pi z}\, dz \tag{38}$$

around a small closed curve enclosing the origin.

Write the integrand as

$$\frac{1+2z^2}{\sin \pi z} = \frac{1}{z}\left[\frac{(1+2z^2)z}{\sin \pi z}\right] \tag{39}$$

where the square bracket is analytic at $z = 0$. We simply evaluate the square bracket in the limit $z \to 0$ to get

$$\begin{aligned} \text{Res}\,(0) &= \pi^{-1} \\ I &= 2i \end{aligned} \tag{40}$$

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Problem 4
Find the same integral I but for a path that encloses the two points $z = 0, 1$. §

A useful formula
Consider a function of the form

$$f(z) = \frac{g(z)}{(z-\zeta)^{n+1}} \tag{41}$$

where $n \geq 0$ and $g(z)$ is analytic. If we expand g in a Taylor series, it is obvious that the residue of f is

$$\text{Res}\,(\zeta) = \frac{1}{n!}\, g^{(n)}(\zeta) \tag{42}$$

Problem 5
Evaluate the integral

$$\oint \frac{e^{az}}{z^{12}}\, dz \tag{43}$$

for a loop around the origin. §

2.4 Cauchy integral formula

From the above, we get an important result

$$f(\zeta) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-\zeta}\, dz \tag{44}$$

where γ is any closed contour that goes around ζ once, and f is analytic on γ and its interior. The proof is very simple: The only pole of the integrand is at $z = \zeta$, with the residue given by $f(\zeta)$.

But this formula has some powerful and perhaps surprising consequences.

- An analytic function on a domain D is completely determined by the values on the boundary of D. Roughly speaking, the value at ζ is some average of the values on the boundary, with a weight inversely proportional to the distance. This is made more precise in a special case in Problem 6.
- Remember that an analytic function is defined as having a *first* derivative. But the formula (44) can be differentiated with respect to ζ any number of time, giving

  $$f^{(n)}(\zeta) = \frac{n!}{2\pi i}\oint_\gamma \frac{f(z)}{(z-\zeta)^{n+1}}\, dz \tag{45}$$

  Thus, f is infinitely differentiable.

Problem 6
As a simple application of the Cauchy integral formula, show that the value of a harmonic function u at any point on the x–y plane is exactly equal to the average value on any circle around that point. Without loss of generality, take the point to be the origin and use polar coordinates: The claim is

$$u(0,0) = \frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)\, d\theta \tag{46}$$

for any $r > 0$. Hint: Take the real part of (44). §

2.5 Roots

Counting roots
Suppose $f(z)$ is an analytic function that has a root at $z = \zeta$. Consider

$$I = \frac{1}{2\pi i}\oint \frac{f'(z)}{f(z)}\, dz \tag{47}$$

around a closed path enclosing the root. Since $f(z)$ is analytic, we can represent it as a power series in $(z-\zeta)$; the zero-order term is absent because ζ is a root. We assume ζ is a simple root. Thus

$$\begin{aligned} f(z) &= a_1(z-\zeta) + a_2(z-\zeta)^2 + ... \\ f'(z) &= a_1 + 2a_2(z-\zeta) + \dots \\ f'(z)/f(z) &= 1/(z-\zeta) + \dots \end{aligned} \tag{48}$$

Note that a_1 cancels and the coefficient in f'/f is exactly 1. So that the residue is exactly 1, and

$$\frac{1}{2\pi i}\oint \frac{f'(z)}{f(z)}\,dz = 1 \quad (49)$$

If the path encloses n simple roots at ζ_j, $j = 1,\ldots,n$, then we get a term like (49) for each pole, and the integral gives the total number of roots inside the contour.

A root of order k counts as k roots for this purpose, since in this case the power series starts at the kth term:

$$\begin{aligned} f(z) &= a_k(z-\zeta)^k + ... \\ f'(z) &= a_k k(z-\zeta)^{k-1} + \ldots \\ f'(z)/f(z) &= k/(z-\zeta) + \ldots \\ I &= k \end{aligned} \quad (50)$$

This gives us a way to count roots inside any contour γ.

Roots of polynomials

We now prove the theorem that an nth order polynomial

$$f(z) = a_n z^n + ... + a_1 z + a_0 \quad (51)$$

with $a_n \neq 0$ has exactly n roots, with high-order roots counted according to the order of the root. To prove this, consider the integral (47) along a circle of radius $R \to \infty$. On this circle, $|z| = R \to \infty$ and only the leading term matters; thus

$$\begin{aligned} f(z) &\approx a_n z^n = a_n R^n e^{in\theta} \\ f'(z) &\approx a_n n z^{n-1} = a_n n R^{n-1} e^{i(n-1)\theta} \\ f'/f &\approx nR^{-1}e^{-i\theta} \\ dz &= d(Re^{i\theta}) = iRe^{i\theta}\,d\theta \\ I &= (2\pi i)^{-1}\int_0^{2\pi} nR^{-1}e^{-i\theta}\cdot iRe^{i\theta}\,d\theta \\ &= n \end{aligned} \quad (52)$$

So a sufficiently large circle encloses exactly n roots. Suppose all the roots are contained within a disk of radius R_1. Somehow we can still "sense" the total number of roots by evaluating the function at $|z| = R \gg R_1$.

3 Vector representation*

**This Section should be skipped if students do not know vector calculus.*

In Part I, it was already mentioned that a complex number $z = x+yi$ can be regarded as the point $P = (x, y)$ on a 2D plane. A differential dz can be associated with the vector

$$d\vec{r} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} \quad (53)$$

Many of the properties discussed above can be put into such vector language.

3.1 Integral and curl

Consider an integral

$$\begin{aligned} I &= \int f(z)\,dz = \int (u+iv)\cdot(dx+idy) \\ &= \int (u\,dx - v\,dy) + i\int (v\,dx + u\,dy) \\ &= \int \vec{\phi}\cdot d\vec{r} + i\int \vec{\psi}\cdot d\vec{r} \end{aligned} \quad (54)$$

where we have introduced vectors

$$\begin{aligned} \vec{\phi} &= u\,\hat{\mathbf{i}} - v\,\hat{\mathbf{j}} \\ \vec{\psi} &= v\,\hat{\mathbf{i}} + u\,\hat{\mathbf{j}} \end{aligned} \quad (55)$$

Note the minus sign arising from i^2.

These integrals around a close loop would be zero if the curls of these vectors are zero. Let us check:[7]

$$\begin{aligned} \text{curl}\,\vec{\phi} &= \frac{\partial\phi_x}{\partial y} - \frac{\partial\phi_y}{\partial x} \\ &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \text{curl}\,\vec{\psi} &= \frac{\partial\psi_x}{\partial y} - \frac{\partial\psi_y}{\partial x} \\ &= \frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} \end{aligned} \quad (56)$$

The vanishing of these curls is just the Cauchy–Riemann conditions (11) and (12). This provides another way to prove the Cauchy integral theorem: that $\oint f(z)\,dz = 0$ if $f(z)$ is analytic everywhere inside the contour.

3.2 Divergence

Next consider the divergence.

[7] In 2D, the curl has only one component.

Problem 5
Show that

$$\begin{aligned} \text{div } \vec{\phi} &= -\text{curl } \vec{\psi} \\ \text{div } \vec{\psi} &= \text{curl } \vec{\phi} \end{aligned} \tag{57}$$

and hence both vanish as well. Note: In 3D, a div (a scalar) cannot possibly be equal to a curl (a 3-vector). But in 2D, a curl has only one component. §

3.3 Laplacian and harmonic property

Since curl $\vec{\phi} = 0$, we can write

$$\vec{\phi} = \vec{\nabla}\Phi \tag{58}$$

for some scalar Φ. Then

$$\nabla^2\Phi = \text{div } \vec{\phi} = 0 \tag{59}$$

Thus

$$\begin{aligned} \nabla^2\vec{\phi} &= \nabla^2(\vec{\nabla}\Phi) \\ &= \vec{\nabla}\,(\nabla^2\Phi) = 0 \end{aligned} \tag{60}$$

which implies

$$\nabla^2 u = \nabla^2 v = 0 \tag{61}$$

recovering the harmonic property in another way.

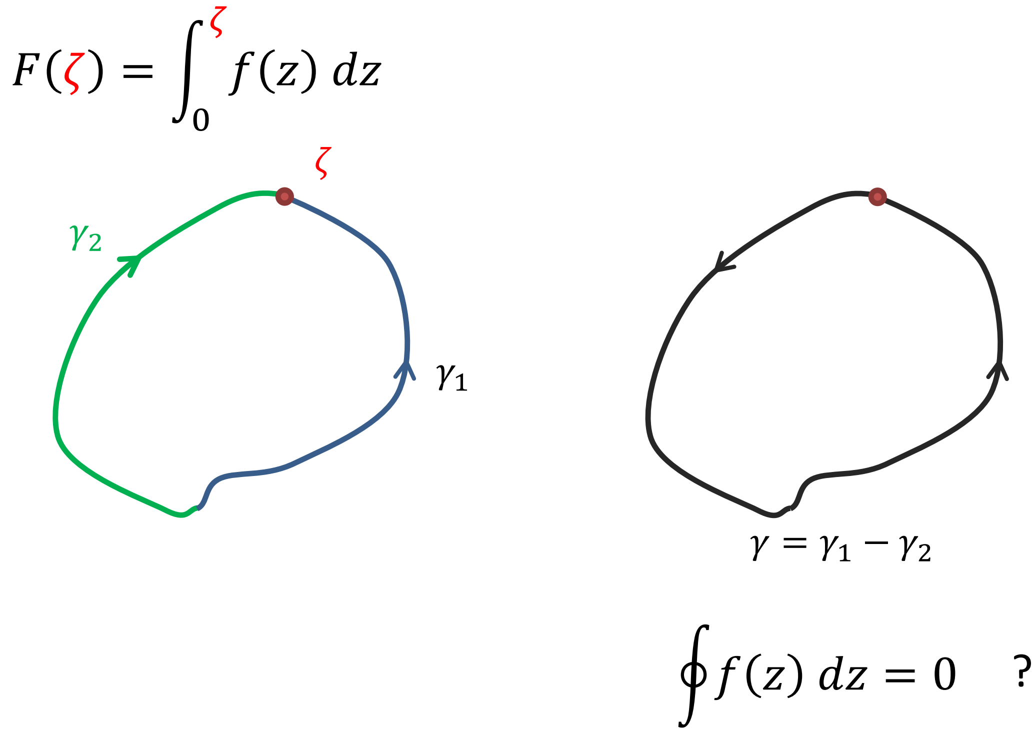

Figure 1: Is the integral along γ_1 and γ_2 the same? That is same as asking whether the integral around the closed loop $\gamma = \gamma_1 - \gamma_2$ is zero.

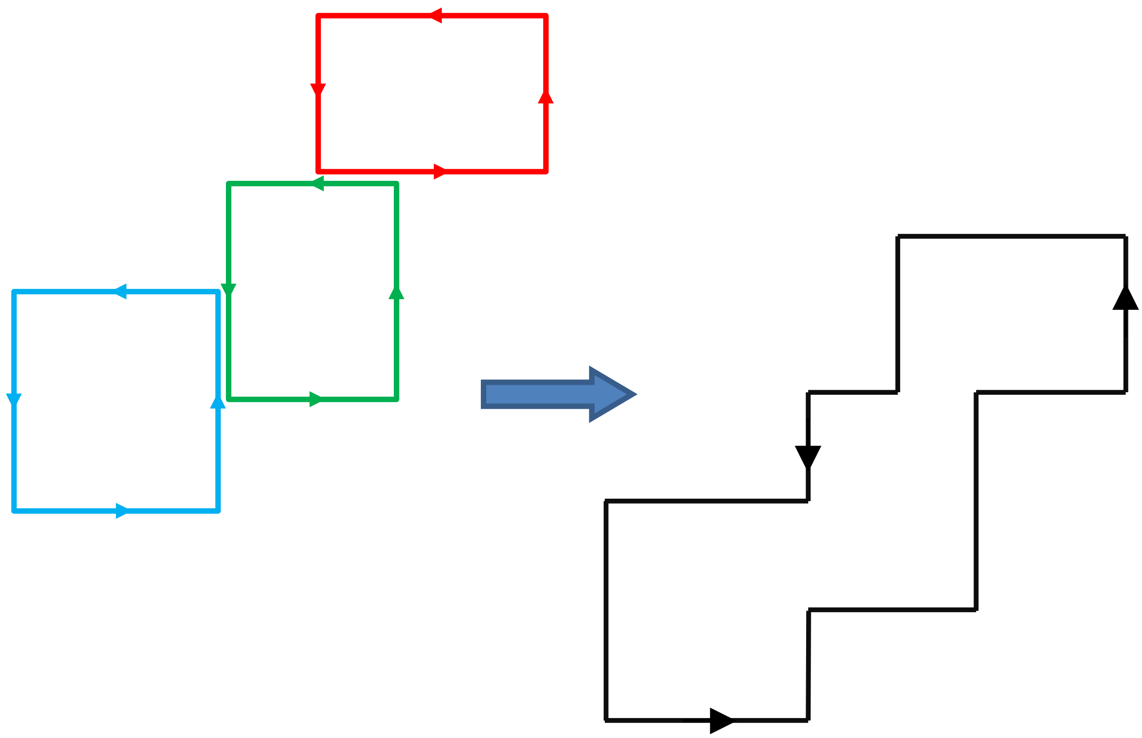

Figure 2: If the integral around any small closed rectangle is zero, then the integral around any closed curve is zero.

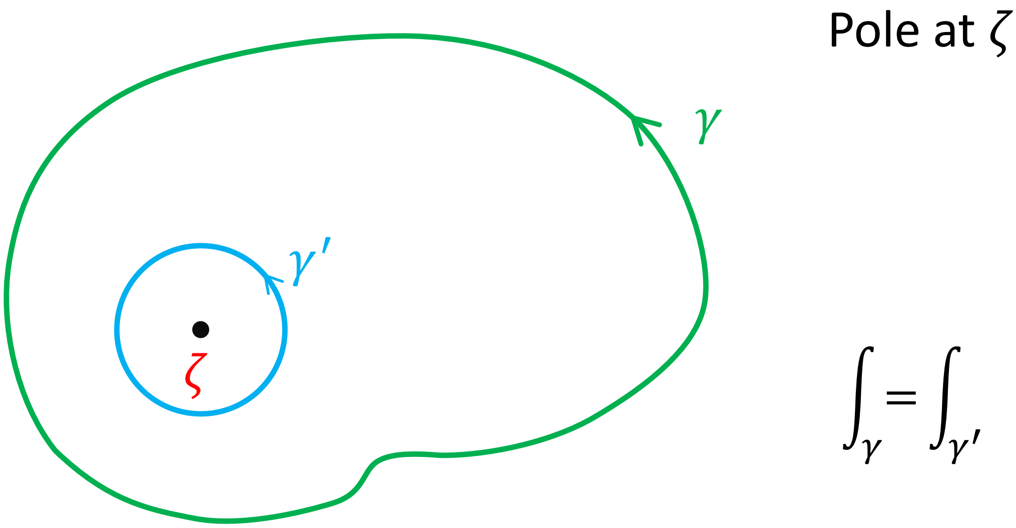

Figure 3: If there is only a pole at ζ, then the integral along γ and along γ' are the same. See the next figure for reason

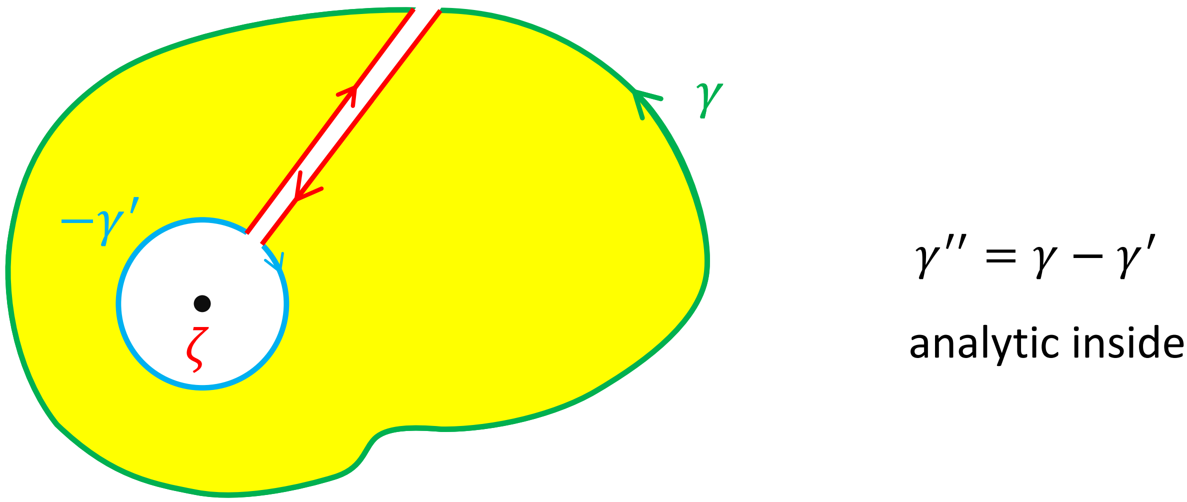

Figure 4: The integral around γ'' is zero, because the function is analytic everywhere inside this closed curve. The contour γ'' is the same as $\gamma - \gamma'$ because the two red lines cancel.