Dimensional analysis: Part I

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This module introduces some elementary aspects of dimensional analysis. Part II deals with several more advanced topics.

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1 Units and dimensions

1.1 Fundamental units

Every physical quantity must carry a unit, e.g. a coordinate or length x in meters (m), a mass m in kilogram (kg), a time t in seconds (s) and a current I in amperes (A); see first two columns in Table 1. These are the four fundamental units[1] in the International System of Units (SI, abbreviation based on French). In mechanics, if currents and charges are not involved, one also speaks of the first three as the MKS system.

quantity	unit	dimension
length x	m	L
mass m	kg	M
time t	s	T
current I	A	I

Table 1. Fundamental units and dimensions

[1] We skip temperature, which is really just energy.

A symbol such as x denotes a physical quantity including its units. Thus we may have, for example

$$x = 3.0 \text{ m} \tag{1}$$

which is a product of a pure number and a unit.

Incidentally, the normal convention is that physical quantities are denoted in italics (e.g. x) and units are in roman type (e.g. m).

1.2 Compound units

Compound units are built up in the natural way. For example, since velocity is defined as

$$v = \frac{\Delta x}{\Delta t} \tag{2}$$

its unit is m s^{-1}.

We use square brackets to denote "the unit of". Thus for x, v, the acceleration a and the force $F = ma$, we have

$$\begin{aligned} [x] &= \text{m} \\ [v] &= \text{m s}^{-1} \\ [a] &= \text{m s}^{-2} \\ [F] &= [m]\,[a] = \text{kg m s}^{-2} \end{aligned} \tag{3}$$

and the derived unit newton (N) is defined as

$$\text{N} = \text{kg m s}^{-2} \tag{4}$$

Therefore every physical quantity X must have a unit given by four indices:

$$[X] = \text{kg}^{\alpha}\ \text{m}^{\beta}\ \text{s}^{\gamma}\ \text{A}^{\delta} \tag{5}$$

The last factor is not necessary in mechanics if electricity and magnetism are not involved.

The special case of all indices being zero, i.e., the pure number 1, is also a unit. The simplest example is angle, measured in radians (rad), which is the ratio of two lengths (arc and radius).

1.3 Dimensions

Of course one can use other units, e.g., lengths in inches (in). Thus, to be more general and not be tied down to SI units, one also refers to the concept of *dimension*, in terms of length (L), mass (M), time (T) and current (I), as shown in the rightmost column in Table 1. If we also use square brackets to denote dimensions, we can say, for example.

$$[F] \quad = \quad \mathrm{M\,L\,T^{-2}} \tag{6}$$

For all practical purposes we can ignore the difference between the dimension and the SI unit for that dimension.

2 Consistency and conversion of units

2.1 Consistency in equations

Example 1
Consider the law of conservation of energy expressed as an equation

$$U + K \quad = \quad E \tag{7}$$

where U, K, E are respectively the potential energy, the kinetic energy and the total energy. The three terms must have the same dimensions, and indeed must be expressed in the same units if we are going to add and subtract.

Potential energy U is related to some work done[2] W, which is in turn defined as some product of force F and displacement Δx. Kinetic energy is, in obvious notation, $K = (1/2)mv^2$. So let us check that the units are correct.

$$\begin{aligned} [U] &= [W] = [F]\,[x] \\ &= \mathrm{kg\ m\ s^{-2}} \cdot \mathrm{m} = \mathrm{kg\ m^2\ s^{-2}} \\ [K] &= [m]\,[v]^2 \\ &= \mathrm{kg} \cdot \left(\mathrm{m\ s^{-1}}\right)^2 \\ &= \mathrm{kg\ m^2\ s^{-2}} \end{aligned} \tag{8}$$

showing consistency between the two terms. §

[2] For the purpose here we need not worry about signs etc.

2.2 Showing units in intermediate steps

The best way to ensure consistency of units is to show them in all intermediate steps — although, with experience, you can omit them *provided* you first express everything in SI units, e.g., no mixture of meter and centimeter, seconds and hours etc.

Example 2
Here is a trivial example: Calculate the distance x covered by a car travelling at $v = 120$ m s^{-1} in $t = 200$ s.

$$\begin{aligned} x &= vt \\ &= 120\ \mathrm{m\ s^{-1}} \cdot 200\ \mathrm{s} \\ &= (120 \times 200) \cdot \left(\mathrm{m\ s^{-1}\ s}\right) \\ &= \left(2.4 \times 10^4\right) \cdot \left(\mathrm{m}\ \not{\mathrm{s}}^{-1}\ \not{\mathrm{s}}\right) \\ &= 2.4 \times 10^4\ \mathrm{m} \end{aligned} \tag{9}$$

Notice how (a) the numerical values and the units are handled separately and (b) the units are simplified systematically. §

2.3 Conversion between units

Example 3
What is the distance x covered by a car travelling at $v = 60$ km h^{-1} in $t = 4.0$ min?

The commonest way is to convert all quantities to SI units first:

$$\begin{aligned} v &= 60 \times (1\ \mathrm{km})/(1\ \mathrm{h}) \\ &= 60 \times (10^3\ \mathrm{m})/(3.6 \times 10^3\ \mathrm{s}) \\ &= 16.7\ \mathrm{m\ s^{-1}} \\ t &= 4.0\ \mathrm{min} = 4.0 \times 60\ \mathrm{s} \\ &= 240\ \mathrm{s} \\ x &= vt = \left(16.7\ \mathrm{m\ s^{-1}}\right) \cdot (240\ \mathrm{s}) \\ &= 4 \times 10^3\ \mathrm{m} = 4\ \mathrm{km} \end{aligned} \tag{10}$$

Another way is to first proceed with mixed units:

$$\begin{aligned} x &= vt = 60\,\frac{\mathrm{km}}{\mathrm{h}} \times 4.0\,\mathrm{min} \\ &= 240\,\frac{\mathrm{km\ min}}{\mathrm{h}} \end{aligned} \tag{11}$$

Then multiply by the following factors:

$$1 \quad = \quad \frac{1000\ \mathrm{m}}{1\ \mathrm{km}}$$

$$
\begin{aligned}
1 &= \frac{60 \text{ s}}{1 \text{ min}} \\
1 &= \frac{1 \text{ h}}{3600 \text{ s}}
\end{aligned} \qquad (12)
$$

The arithmetic is left for you to check. Here we just emphasize that this trick of multiplying by unity as a way of converting between units. §

Problem 1
(a) A star X is at a certain distance s. Suppose we move the observation point in a direction perpendicular to the line of sight by 1.00 AU (1 AU = 149.6 million km), and it is found that the direction of the line of sight changes by 1.00 sec of arc. What is the distance s? This distance is called a parsec (pc). Express pc in light years (ly). The speed of light is 3.00×10^8 m s^{-1}.
(b) The speeds v of galaxies away from us and the distances s away from us are found to be proportional: $v = Hs$, where H is called Hubble's constant. Because v is measured in units of km s^{-1}, and s is measured in Mpc, the value of H is usually given in units of km s^{-1} Mpc^{-1}. Recent data show that $H \approx 70$ in these units. Express H^{-1} in units of seconds.
(c) To a good approximation, H^{-1} is the age of the universe. (Can you understand why?) From the above data, find the age of the universe in Gy. §

3 Dimensional analysis

The idea of dimensional analysis can be illustrated by an example.

Example 4
A mass m is attached to the end of a spring obeying Hooke's law: $F = -kx$, where F is the force acting on the mass and x is the displacement from the equilibrium position. The minus sign indicates that the force is opposite to the displacement, and the proportionality constant k is called the force constant of the spring. This mass is pulled from equilibrium, and goes into oscillations with amplitude A. What can be said about the period T of oscillations?

Let us suppose that T is given by

$$
T = C\, m^\alpha k^\beta A^\gamma \qquad (13)
$$

where C is a pure number.

First we note that

$$
\begin{aligned}
[F] &= \text{kg m s}^{-2} \\
[k] &= [F]\,[x]^{-1} \\
&= \text{kg s}^{-2}
\end{aligned} \qquad (14)
$$

Then, by comparing units on both sides of (13), we find

$$
\begin{aligned}
\text{s} &= \text{kg}^\alpha \left(\text{kg s}^{-2}\right)^\beta \text{m}^\gamma \\
&= \text{kg}^{\alpha+\beta}\, \text{m}^\gamma\, \text{s}^{-2\beta}
\end{aligned} \qquad (15)
$$

Matching powers of kg, m and s, we find

$$
\begin{aligned}
0 &= \alpha + \beta \\
0 &= \gamma \\
1 &= -2\beta
\end{aligned} \qquad (16)
$$

or

$$
(\alpha, \beta, \gamma) = (1/2, -1/2, 0) \qquad (17)
$$

so that

$$
T = C\sqrt{m/k} \qquad (18)
$$

independent of the amplitude A. §

The constant C cannot be determined this way — lunch is not completely free. But even without knowing its value, we have made considerable advances. (a) The dependence on the parameters is known, e.g., that the period is independent of amplitude. Incidentally, students should develop the habit of checking intuitively whether the direction of dependence makes sense: if the mass is heavier ($m \uparrow$) or if the spring is softer ($k \downarrow$), then the period is longer ($T \uparrow$). (b) If we do a single measurement or a single numerical solution (i.e., for one set of m, k, A), we would know the answer for all parameter values. (c) Even without any measurement or numerical solution, we can guess that C cannot be very large (e.g. 10^3) or very small (e.g. 10^{-3}), so we have an order-of-magnitude estimate without any further work. In this particular case, it turns out that there is an analytic solution, and $C = 2\pi$, but this result is beyond the present topic of dimensional analysis.

However, there is an implicit assumption: that no other physical quantity is relevant, e.g. the mass M of the moon, the speed of light c or the period of the earth's rotation T_E. If any such variables are involved, then (13) would look more complicated.

Problem 2
With reference to the above example, explain the following.
(a) Why is it not allowed for C to have units, for example seconds? (Hint: the paragraph immediately above.)
(b) On the RHS of (13) we have assumed simple powers. Why is it not allowed to have more complicated dependence, such as $\sin a\xi$ or $\exp(-a\xi)$, where a is a pure number and ξ is some combination of m, k and A? §

Problem 3
Suppose we have a spring that satisfies $F = -kx^3$.
(a) Find the unit of k.
(b) If the period T is again given by the formula (13), find the indices α, β and γ.
(c) If you have already learnt numerical methods, determine the constant C in this case. §

Problem 4
A pendulum consists of a mass m tied to the end of a string of length ℓ. The pendulum swings on account of gravity, whose strength is described by the acceleration due to gravity g (i.e. the gravitational force per unit mass). For small amplitudes, the period is independent of amplitude and is given by

$$T = C\, m^\alpha g^\beta \ell^\gamma \tag{19}$$

Find the indices α, β and γ. (It turns out that the constant is again $C = 2\pi$.) §

Problem 5
An aeroplane with total wing area A is flying at speed v through air of density ρ. The lifting force L by the air on the wings is given by

$$L = (C/2) \cdot \rho^\alpha\, v^\beta\, A^\gamma \tag{20}$$

where the factor of 2 is a matter of convention and the pure number C is called the lift coefficient. We assume all dimensionless parameters (e.g., the width-to-length ratio of the wings, the angle by which the nose of the plane is tilted upwards, which is called the angle of attack) are fixed; otherwise there could be complicated dependence on these parameters.
(a) Using dimensional analysis, determine the indices α, β, γ.
(b) By performing measurements on a small model Airbus A340 in a wind tunnel (thus much smaller A, smaller v and approximately the same ρ), it was determined that $C = 1.5$ when the plane is in the takeoff configuration. The total mass of the actual A340 is 2.6×10^5 kg and the wing area is $2\times 360\ \mathrm{m}^2$. What speed must be achieved at take-off? Express your answer in km h^{-1}. §