THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050B Mathematical Analysis I (Fall 2016)
Homework 3 Suggested Solutions to Starred Questions

1(a). (For the tutorial on 6 Oct) Show (without use of ratio test) that

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0$$

Hint: If $2K < n$ then

$$\frac{n!}{n^n} < \frac{K(K-1)\cdots 3\cdot 2\cdot 1}{n\cdot n\cdots n\cdot n} < \left(\frac{1}{2}\right)^K$$

Proof. If $2K < n$, then

$$\begin{aligned}
\frac{n!}{n^n} &= \frac{n(n-1)\cdots(2K+1)}{n^{n-2K}} \cdot \frac{2K(2K-1)(2K-2)\cdots 2\cdot 1}{n^{2K}} \\
&< \frac{2K(2K-1)(2K-2)\cdots 2\cdot 1}{n^{2K}} \\
&= \frac{2K(2K-1)\cdots(K+1)}{n^K} \cdot \frac{K(K-1)(K-2)\cdots 2\cdot 1}{n^K} \\
&< \frac{K(K-1)(K-2)\cdots 2\cdot 1}{n^K} \\
&< \frac{1}{2}\cdot\frac{1}{2}\cdots\frac{1}{2} \\
&= \left(\frac{1}{2}\right)^K
\end{aligned}$$

Let $\epsilon > 0$. Using the fact that $\lim_{k\to\infty}\left(\frac{1}{2}\right)^k = 0$, there exists $N \in \mathbb{N}$ such that for all $k \geq N$,

$$\left(\frac{1}{2}\right)^k < \epsilon$$

In particular,

$$\left(\frac{1}{2}\right)^N < \epsilon$$

Now for $n > 2N$, we have, with K replaced by N, that

$$\frac{n!}{n^n} < \left(\frac{1}{2}\right)^N < \epsilon$$

Hence

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0$$

□

1(b). Let $b > 0$. Show that $b^n << n!$ in the sense that

$$\lim_{n\to\infty} \frac{b^n}{n!} = 0$$

Proof. We will apply the ratio test. If $a_n := \frac{b^n}{n!} > 0$, then

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{b^{n+1}}{(n+1)!}}{\frac{b^n}{n!}} = \lim_{n\to\infty} \frac{b}{n+1} = 0$$

Hence $\frac{b^n}{n!}$ converges to 0. □

2(a). Show that

$$\lim_{n\to\infty} n^{\frac{1}{n}} = 1.$$

Proof. Let $\epsilon > 0$. Consider the inequality

$$|n^{\frac{1}{n}} - 1| < \epsilon$$

which, by algebraic manipulation, splits into two inequalities which must be satisfied simultaneously:

$$n > (1-\epsilon)^n \tag{1}$$

$$n < (1+\epsilon)^n \tag{2}$$

Here (1) is trivial, since $(1-\epsilon)^n < 1 \leq n$.

Next we consider (2): By binomial theorem, we have:

$$(1+\epsilon)^n = \sum_{k=0}^{n} \binom{n}{k} \epsilon^k \geq \binom{n}{2} \epsilon^2 = \frac{n(n-1)}{2}\epsilon^2,$$

if $n > 2$.

Hence we choose $N > 1 + \frac{2}{\epsilon^2}$ by Archimedean Property. By our choice of N, we have: for $n \geq N$,

$$\frac{n(n-1)}{2}\epsilon^2 \geq \frac{n(N-1)}{2}\epsilon^2 > \frac{n[(1+\frac{2}{\epsilon^2})-1]}{2}\epsilon^2 = n,$$

which, combined with the above estimate, shows that for $n \geq N$,

$$n < (1+\epsilon)^n$$

Combining (1)(2) we have that for $n \geq N$,

$$|n^{\frac{1}{n}} - 1| < \epsilon$$

Hence

$$\lim_{n\to\infty} n^{\frac{1}{n}} = 1.$$

□

3. (Together with the tutorial)

Definition 1. *Let* (x_n) *be any sequence of real numbers. We define its partial sum*

$$S_n := \sum_{k=1}^{n} x_k,$$

and then the average of it by

$$A_n := \frac{S_n}{n}.$$

(a) Show that if $\lim_{n\to\infty} x_n = x \in \mathbb{R}$, then

$$\lim_{n\to\infty} A_n = x.$$

(b) (Tutorial Question)Show that the converse of (a) is not true by constructing a real sequence a_n whose average converges to a finite limit $l \in \mathbb{R}$ but a_n itself diverges.

(a) *Proof.* For simplicity of notations, we may first assume that $x = 0$.
Let $\epsilon > 0$. Since x_n converges to $x = 0$, there is $N_1 \in \mathbb{N}$ such that for $n \geq N_1$, $|x_n| < \frac{\epsilon}{2}$.
By Archimedean Property, let $N_2 \in \mathbb{N}$ such that $N_2 > \frac{2|S_{N_1}|}{\epsilon}$. Then for $n \geq N_2$, we have:

$$\begin{aligned}
\left|\frac{x_1 + x_2 + \cdots + x_n}{n}\right| &= \frac{1}{n}\left|x_1 + \cdots + x_{N_1} + x_{N_1+1} + \cdots + x_n\right| \\
&\leq \frac{1}{n}\left(|x_1 + \cdots + x_{N_1}| + |x_{N_1+1}| + \cdots + |x_n|\right) \\
&\leq \frac{1}{n}\left(|S_{N_1}| + (n - N_1)\frac{\epsilon}{2}\right) \\
&\leq \frac{|S_{N_1}|}{n} + \frac{\epsilon}{2} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

This proves the case when $x = 0$.
Now we consider the general case. Let $(x_n) \to x \in \mathbb{R}$. Define $y_n := x_n - x$, then $y_n \to 0$. Applying the above result to y_n, we see that

$$\lim_{n\to\infty} \frac{y_1 + y_2 + \cdots + y_n}{n} = 0$$

But

$$\frac{y_1 + y_2 + \cdots + y_n}{n} = \frac{x_1 - x + x_2 - x + \cdots + x_n - x}{n} = \frac{x_1 + x_2 + \cdots + x_n}{n} - x.$$

Hence by computation rules of limits, we have:

$$\lim_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n} = \lim_{n\to\infty} \frac{y_1 + y_2 + \cdots + y_n}{n} + x = x.$$

□

(b) Let (x_n) be defined by $x_n := (-1)^n$. Then (x_n) diverges, but its average is computed to be:

$$A_n := \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{(-1)^n - 1}{2n}$$

Noting that

$$-\frac{1}{n} \leq A_n \leq 0,$$

By squeeze law, we have $\lim_{n\to\infty} A_n = 0$.