THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050B Mathematical Analysis I (Fall 2016)
Homework 1 Suggested Solutions to Starred Questions

1.(a) $a \cdot 0 = 0$.

Proof.

$$\begin{aligned} a \cdot 0 &= a \cdot (0 + 0) && \text{(definition of 0)} \\ &= a \cdot 0 + a \cdot 0 && \text{(distributive law)} \end{aligned}$$

Adding $-(a \cdot 0)$ on both sides on the left (or right if you like), we have:

$$\begin{aligned} -(a \cdot 0) + a \cdot 0 &= -(a \cdot 0) + (a \cdot 0 + a \cdot 0) \\ -(a \cdot 0) + a \cdot 0 &= [-(a \cdot 0) + a \cdot 0] + a \cdot 0 && \text{(associativity of addition)} \\ 0 &= 0 + a \cdot 0 && \text{(definition of additive inverse)} \\ 0 &= a \cdot 0 && \text{(definition of 0)} \end{aligned}$$

□

1.(b) $-a = (-1) \cdot a$.

Proof. We will use the fact that additive inverses are unique in the following sense: Let $a \in \mathbb{R}$. Suppose there is some real number b such that

$$a + b = b + a = 0,$$

then $b = (-a)$, the additive inverse of a.

Now by the above, the definition of $-a$ and commutativity of addition, it suffices to show that

$$(-1) \cdot a + a = 0$$

which is done as follows:

$$\begin{aligned} (-1) \cdot a + a &= (-1) \cdot a + 1 \cdot a && \text{(definition of 1)} \\ &= [(-1) + 1] \cdot a && \text{(distributive law)} \\ &= 0 \cdot a && \text{(definition of additive inverse)} \\ &= 0 && \text{(by 1(a))} \end{aligned}$$

□

2(a) Show that $|x-a|<\epsilon$ if and only if

$$a-\epsilon<x<a+\epsilon$$

Remark: We can assume that $\epsilon>0$.

Proof. " $\Longrightarrow$ " Suppose $|x-a|<\epsilon$ (Thus $\epsilon>0$).
Recall that

$$|x-a|=\begin{cases} x-a, & \text{if } x>a \\ a-x, & \text{if } x\leq a \end{cases}$$

Hence we have: $(x>a$ and $x-a<\epsilon)$ or $(x\leq a$ and $-x+a<\epsilon)$

By calculation, this gives: $(a<x<a+\epsilon)$ or $(a-\epsilon<x\leq a)$

Hence we have $a-\epsilon<x<a+\epsilon$.

" $\Longleftarrow$ " Suppose $a-\epsilon<x<a+\epsilon$ (Thus $\epsilon>0$). Then adding $-a$ on both sides,

$$-\epsilon<x-a<\epsilon.$$

Hence

$$(-\epsilon<x-a<\epsilon \text{ and } x-a\geq 0) \text{ or } (-\epsilon<x-a<\epsilon \text{ and } x-a<0),$$

namely

$$(x-a<\epsilon \text{ and } x-a\geq 0) \text{ or } (-x+a<\epsilon \text{ and } x-a<0).$$

Then,

$$(|x-a|<\epsilon \text{ and } x-a\geq 0) \text{ or } (|x-a|<\epsilon \text{ and } x-a<0),$$

and hence

$$|x-a|<\epsilon.$$

□

3(a) Let A be a nonempty subset of real numbers and $l\in\mathbb{R}$. State the definition and the negation for the following:

l is a lower bound of A.

Solution:

Definiton: Let A be a nonempty subset of real numbers and $l\in\mathbb{R}$. We say l is a lower bound of A if for each $a\in A$, we have $a\geq l$.

Negation: Let A be a nonempty subset of real numbers and $l\in\mathbb{R}$. We say l is NOT a lower bound of A if there exists $a\in A$ such that $a<l$.

4(b) Let $(x_n),(y_n)$ be sequences of real numbers converging to $x,y\in\mathbb{R}$ respectively. Show that

$$\lim_{n\to\infty}(x_n+y_n)=x+y.$$

Proof. Let $\epsilon > 0$. Since x_n converges to $x \in \mathbb{R}$, there exists $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, we have

$$|x_n - x| < \frac{\epsilon}{2}.$$

Similarly, since y_n converges to $y \in \mathbb{R}$, there exists $N_2 \in \mathbb{N}$ such that for any $n \geq N_2$, we have

$$|y_n - y| < \frac{\epsilon}{2}.$$

We take $N = \max\{N_1, N_2\}$. For $n \geq N$, both inequalities above are satisfied, thus by triangle inequality,

$$|x_n + y_n - x - y| \leq |x_n - x| + |y_n - y| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $(x_n + y_n)$ converges to $x + y$.

Remark: The numbers $\frac{\epsilon}{2}$ are chosen so as to make the final sum add to ϵ, which is what we want ultimately. Similarly, if we had 3 or more, say, m such inequalities which is needed to sum to ϵ, since ϵ is arbitrary, we would then take each number in the inequalities as $\epsilon/3, \epsilon/m$, etc. $\square$