MATH1010 Revision Exercise 1

1. Determine, for each of the limits below, whether it exists or not. Give appropriate justification. Where the limit exists, also evaluate its value.

(a) $\lim_{x\to 0} \frac{x^2-1}{2x^2-x+1}$ (b) $\lim_{x\to 0} \frac{x-1}{\sqrt{x^2+3}-2}$ (c) $\lim_{x\to 0+} \frac{1+\sqrt{x}}{1-\sqrt{x}}$ (d) $\lim_{x\to 2} \frac{\sqrt{2+x}-2}{x-2}$

(e) $\lim_{x\to 1^+} \frac{1-x}{\sqrt{x-1}}$ (f) $\lim_{x\to 1} \frac{x^2-1}{3x^2-x-2}$ (g) $\lim_{x\to 3} \frac{x^2-5x+6}{x^2+2x-15}$ (h) $\lim_{x\to 1} \frac{x^2-3x+2}{x^2+5x-6}$

(i) $\lim_{x\to 0} \frac{(1+x)^4-(1+4x)}{x+x^4}$ (j) $\lim_{x\to 0} \frac{(1+2x)(1+3x)(1+4x)-1}{x}$ (k) $\lim_{x\to 2} \frac{2-x}{3-\sqrt{x^2+5}}$

(l) $\lim_{x\to \frac{1}{2}} \left[\frac{2x^2-1}{(3x+2)(5x-3)} - \frac{2-3x}{x^2-5x+3}\right]$ (m) $\lim_{x\to 1} \frac{1}{x-1}\left(\frac{1}{x+3}-\frac{2x}{3x+5}\right)$

2. (a) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

i. Show that f is continuous at 0.

ii. Is f differentiable at 0? If it is, also compute $f'(0)$. Justify your answer.

(b) Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by

$$g(x) = \begin{cases} \frac{e^x-1}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

i. Show that g is continuous at 0.

ii. Is g differentiable at 0? If it is, also compute $f'(0)$. Justify your answer.

(c) Let $h : (-1, +\infty) \longrightarrow \mathbb{R}$ be the function defined by

$$h(x) = \begin{cases} \frac{\ln(1+x)}{x} & \text{if } x > -1 \text{ and } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

i. Show that h is continuous at 0.

ii. Is h differentiable at 0? If it is, also compute $f'(0)$. Justify your answer.

Remark. You may be inadvertently using the continuity of the functions f, g, h at 0 in the next two questions, when you are considering limits of functions involing the exponential function, the logarithmic function and the trigonometric functions.

3. Determine, for each of the limits below, whether it exists or not. Give appropriate justification. Where it exists, also evaluate the value of the limit.

(a) $\lim_{x\to 0} \frac{1-2\cos(x)+\cos(2x)}{x^2}$ (b) $\lim_{x\to 0} \frac{\sin(3x)}{x}$ (c) $\lim_{x\to 0} \frac{\sin(5x)}{3x}$ (d) $\lim_{x\to 0} \frac{\sin(3x)}{\sin(2x)}$

(e) $\lim_{x\to 0} \frac{\cos(5x)-\cos(x)}{x^2}$ (f) $\lim_{x\to 0} \frac{\sin(x^2)}{x}$ (g) $\lim_{x\to 0} \frac{\sec(x)-\cos(x)}{\sin(x)}$ (h) $\lim_{x\to 1^+} \frac{1+\cos(\pi x)}{\tan^2(\pi x)}$

(i) $\lim_{x\to \frac{\pi}{2}} (\sec(x)-\tan(x))$ (j) $\lim_{x\to 0} \frac{6x-\sin(2x)}{2x+3\sin(4x)}$ (k) $\lim_{x\to 0} \frac{\tan(x)-\sin(x)}{x^3}$

(l) $\lim_{x\to\frac{\pi}{2}} \left(x - \frac{\pi}{2}\right)\sec(x)$ (m) $\lim_{x\to 3}(x-3)\csc(\pi x)$ (n) $\lim_{x\to 0} \frac{1-2\cos(x)+\cos(2x)}{x^2}$

(o) $\lim_{x\to 1} \frac{3\sin(\pi x) - \sin(3\pi x)}{x^3}$ (p) $\lim_{x\to 0} \frac{\cos(ax)-\cos(bx)}{x^2}$. (Here a, b are real numbers.)

4. Determine, for each of the limits below, whether it exists or not. Give appropriate justification. Where it exists, also evaluate the value of the limit.

(a) $\lim_{x\to 0} \frac{4^x - 4^{-x}}{4^x + 4^{-x}}$ (b) $\lim_{x\to 0} \frac{e^{-ax} - e^{-bx}}{x}$. (Here a, b are real numbers.)

(c) $\lim_{x\to 0} \frac{a^x - b^x}{x}$. (Here a, b are positive real numbers.) (d) $\lim_{x\to 0} \frac{\tanh(ax)}{x}$. (Here a is a real number.)

5. Determine, for each of the limits below, whether it exists or not. Give appropriate justification. Where the limit exists, also evaluate its value.

(a) $\lim_{x\to+\infty} \frac{(3x-1)(2x+3)}{(5x-3)(4x+5)}$ (b) $\lim_{x\to-\infty} \left(\frac{3x}{x-1} - \frac{2x}{x+1}\right)$ (c) $\lim_{x\to+\infty} (\sqrt{x+1} - \sqrt{x})$

(d) $\lim_{x\to+\infty} \frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt{1+x}}$ (e) $\lim_{x\to+\infty} \frac{2x^4 - 3x^2 + 1}{6x^4 - x^3 - 3x}$ (f) $\lim_{x\to+\infty} \frac{x^4 - 3x^2}{x^5+1}$

(g) $\lim_{x\to+\infty} \left(\sqrt{x^4+1} - x^2\right)$ (h) $\lim_{x\to+\infty} \sqrt{x+2}\,(\sqrt{x+1} - \sqrt{x})$

(i) $\lim_{x\to+\infty} x\left(\sqrt{x^2+2x} - 2\sqrt{x^2+x} + x\right)$ (j) $\lim_{x\to-\infty} \left(\sqrt{x^2+1} + x\right)$

(k) $\lim_{x\to+\infty} \frac{x+\cos(x)}{x+1}$ (l) $\lim_{x\to+\infty} e^{-x}\sin(x)$ (m) $\lim_{x\to+\infty} \frac{2^x - 2^{-x}}{2^x + 2^{-x}}$

6. Let m, n be positive integers. Does the limit $\lim_{x\to 0} \frac{(1+nx)^m - (1+mx)^n}{x^2 - x}$ exist? Justify your answer. If it exists, also evaluate its value.

7. Let a, ℓ be real numbers. Suppose that the limit $\lim_{x\to -1} \frac{x^3 - ax^2 - x + 4}{x+1}$ exists and is ℓ.

Determine the values of a, ℓ respectively. Justify your answer.

8. Compute the first derivative of each of the functions below:

(a) $\frac{x}{(1+x)^2}$ (b) $\frac{3x+4}{x^2+1}$ (c) $\frac{x}{x^2+1}$ (d) $\frac{2x+1}{x(x^2+1)}$ (e) $\frac{5x-1}{x^2-x+1}$

(f) $\left(\frac{1+x}{1-x}\right)^5$ (g) $\frac{x^3}{(x^2-3)^3}$ (h) $\frac{1}{1-\sqrt{x}}$ (i) $\frac{1+\sqrt{x}}{1-\sqrt{x}}$

(j) $\frac{x+1}{\sqrt{2x^2+x-3}}$ (k) $x^3\sqrt{1+x^2}$ (l) $(x+1)^{\frac{1}{3}}(x+3)^{\frac{7}{3}}$ (m) $x^3(2+x^2)^{\frac{5}{2}}$

9. Compute the first derivative of each of the functions below:

(a) $\cos^2(x) + \sin(3x)$ (b) $\frac{\sin(x)}{2+\cos(x)}$ (c) $\frac{\sin(x)}{\sin(x)+\cos(x)}$ (d) $\sin^3(x)\cos(3x)$

(e) $\sin\left(\frac{1}{x} + \cos(x)\right)$ (f) $\frac{\cos^5(x)}{\sin(5x)}$ (g) $\cos(\sin(4x))$ (h) $\frac{\sin(x)}{\sqrt{5-4\cos(x)}}$

(i) $\sin(x)\sqrt{\cos(2x)}$ (j) $x\tan(2x)$ (k) $\sqrt{\sec(x^2)}$ (l) $\sqrt{\sin(1-2x)}$

(m) $\sqrt{\cos(6x)}\cdot\csc^3(x)$ (n) $\sin(x(1-2x)^{\frac{1}{3}})$ (o) $\cot(\sqrt{x^3-3x})$

(p) $\sqrt{\sec^3(x)\sec(3x)}$ (q) $\tan(\cos(x^{\frac{1}{3}}))$ (r) $\cot(\tan(x^{\frac{5}{4}}))$

10. Compute the first derivative of each of the functions below:

(a) $e^{\cos(x)}$ (b) xe^{x-x^2} (c) $x^2e^{\sin^2(x)}$ (d) $\ln(\sin(x))$ (e) $\ln(\cos(x))$

(f) $\ln(\sec(x)+\tan(x))$ (g) $\ln(\csc(x)+\cot(x))$ (h) $\ln(1-2\cos(2x))$ (i) $(\sin(x))^x$

(j) $x^{(x^2)}$ (k) $x^2 \cdot 2^x$ (l) $x \cdot \pi^{x^2}$ (m) $(\ln(x))^{\ln(x)}$

(n) $(\sqrt{2})^x \cdot x^{\sqrt{2}}$ (o) $\sinh(2x)\cosh^2(x)$

11. Let a, b be real numbers, and $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x^2 + ax & \text{if} \quad x \geq 0 \\ x + b & \text{if} \quad x < 0 \end{cases}$$

(a) Suppose f is continuous at 0. Determine the value of b. Justify your answer.

(b) Suppose f is differentiable at 0. Determine the values of a, b respectively. Justify your answer.

12. Let a, b be real numbers, and $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x^3 - 2x + 1 & \text{if} \quad x \geq 0 \\ a\sin(x) + b & \text{if} \quad x < 0 \end{cases}$$

(a) Suppose f is continuous at 0. Determine the value of b. Justify your answer.

(b) Suppose f is differentiable at 0. Determine the values of a, b respectively. Justify your answer.

13. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x(1+x) & \text{if} \quad x < -4 \\ -3x & \text{if} \quad -4 \leq x < 5 \\ x^2 - 3x - 20 & \text{if} \quad x \geq 5 \end{cases}$$

(a) Is f continuous at 4? Justify your answer.

(b) Is f differentiable at 4? Justify your answer.

(c) Is f continuous at 5? Justify your answer.

(d) Is f differentiable at 5? Justify your answer.

14. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} \dfrac{1-\cos(x)}{x^2} & \text{if} \quad x \neq 0 \\ \dfrac{1}{2} & \text{if} \quad x = 0 \end{cases}$$

(a) Is f continuous at 0? Justify your answer.

(b) Is f differentiable at 0? Justify your answer.

15. (a) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 4x & \text{if} \quad x < 3 \\ 2x^2 - 6 & \text{if} \quad x \geq 3 \end{cases}$$

Determine whether f is differentiable at 3. If it is, also determine the value of $f'(3)$. Justify your answer.

(b) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x + 2 & \text{if} \quad x < -2 \\ (x+2)^2 & \text{if} \quad x \geq -2 \end{cases}$$

Determine whether f is differentiable at -2. If it is, also determine the value of $f'(-2)$. Justify your answer.

(c) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = |x^3 + 8|$ for any $x \in \mathbb{R}$.

Determine whether f is differentiable at -2. If it is, also determine the value of $f'(-2)$. Justify your answer.