Lecture 4
Relaxation

MATH3220 Operations Research and Logistics
Jan. 20, 2015

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Agenda

1. **Relaxation of a particular problem**
2. **The Lagrange Relaxation**
3. **The general form of the B&B method**

General formulation of an optimization problem

$$
\begin{aligned}
\max \quad & f(x) \\
\text{s.t.} \quad & \mathbf{g(x)} \leq \mathbf{b} \\
& x \in A
\end{aligned}
$$

Relaxing the non-algebraic constraints

$x \in A$ is replaced by a requirement that the variables must belong to a set, say B, which is larger than A, i.e., the relation $A \subseteq B$ must hold.
The relaxation of the problem is

$$\begin{aligned} \max \quad & f(x) \\ \text{s.t.} \quad & \mathbf{g(x)} \leq \mathbf{b} \\ & x \in B \end{aligned}$$

Relaxing the algebraic constraints

The algebraic constraint $\mathbf{g(x)} \leq \mathbf{b}$ is relaxed.
A natural way of doing this is the following:
Assume that there are m inequalities in $\mathbf{g(x)} \leq \mathbf{b}$. Let $\lambda_i \geq 0 \quad (i = 1, \ldots, m)$ be fixed numbers. Then any $x \in A$ satisfying $\mathbf{g(x)} \leq \mathbf{b}$ also satisfies the inequality:

$$\sum_{i=1}^{m} \lambda_i g_i(x) \leq \sum_{i=1}^{m} \lambda_i b_i.$$

The name of the inequality is surrogate constraint. Then, the

relaxation problem is:

$$\begin{array}{ll} \max & f(x) \\ \text{s.t.} & \displaystyle\sum_{i=1}^{m} \lambda_i g_i(x) \leq \sum_{i=1}^{m} \lambda_i b_i \\ & x \in A \end{array}$$

Example 1

$$
\begin{aligned}
\text{Max} \quad & 8x_1 + 11x_2 + 6x_3 + 4x_4 \\
\text{s.t.} \quad & 4x_1 + 3x_2 + 3x_3 + 3x_4 \leq 9 \\
& x_1 + 4x_2 + x_3 \leq 5 \\
& x_j \in \{0, 1\}
\end{aligned}
$$

Example 1

$$\begin{aligned} \text{Max} \quad & 8x_1 + 11x_2 + 6x_3 + 4x_4 \\ \text{s.t.} \quad & 4x_1 + 3x_2 + 3x_3 + 3x_4 \leq 9 \quad (P) \\ & x_1 + 4x_2 + x_3 \leq 5 \\ & x_j \in \{0, 1\} \end{aligned}$$

If $\lambda_1 = \lambda_2 = 1$ then the relaxation obtained in this way is

$$\begin{aligned} \text{Max} \quad & 8x_1 + 11x_2 + 6x_3 + 4x_4 \\ \text{s.t.} \quad & 5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14 \quad (R) \\ & x_j \in \{0, 1\} \end{aligned}$$

The optimal solution to (R) is $(0, 1, 1, 1)$ satisfies the constraint of (P), thus it is also the optimal solution of (P).

Example 2

A region defined by nonlinear boundary surfaces can be approximated by tangent planes.
For example, if the feasible region is the unit circuit which is defined by the inequality:

$$x_1^2 + x_2^2 \leq 1$$

can be approximated by

$$-1 \leq x_1, x_2 \leq 1$$

If the optimal solution on the enlarged region is not in the original feasible region, e.g $(1, 1)$, then a cut must be found which cuts it from the relaxed region but it does cut any part of the original feasible region. It can be done like the inequality:

$$x_1 + x_2 \leq \sqrt{2}$$

Relaxing the objective function

In other cases the difficulty of the problem is caused by the objective function. If it is possible to find an easier function, say $h(x)$, but to obtain an upper bound the condition

$$\forall x \in A : h(x) \geq f(x)$$

must hold. Then the relaxation is

$$\begin{array}{rll} \max & h(x) & \\ \text{s.t.} & \mathbf{g(x)} & \leq \mathbf{b} \\ & x \in A & \end{array}$$

The Lagrange Relaxation

The difficulty of the problem caused by the requirement of both constraints. It is significantly easier to satisfy just one type of constraints. So what about to eliminate one of them?

Assume the number of inequalities in the algebraic constraint is m. Let $\lambda_i \geq 0 (i = 1, \ldots, m)$ be fixed numbers. The Lagrange relaxation of the problem is

$$\begin{array}{ll} \text{Max} & f(x) + \sum_{i=1}^{m} \lambda_i (b_i - g_i(x)) \\ \text{s.t.} & x \in A \end{array}$$

Notice, the objective function penalizes the violation of the constraints, and rewards the savings of resources.
Let (P) refer to the original problem, and (L(λ)) denote the Lagrange relaxation, which reflects that the Lagrange relaxation depends on the choice of λ_i's. λ_i's are called the Lagrange multipliers.

Theorem

Assume that both (P) and ($L(\lambda)$) have optimal solutions. Then for any nonnegative λ_i the inequality

$$v(L(\lambda)) \geq v(P)$$

holds, where $v(S)$ denotes the optimal value of problem S.

Proof.

The statement is that the optimal value of ($L(\lambda)$) is an upper bound of the optimal value of (P). Let x^* be the optimal solution of (P). It is obviously feasible in both problems. Hence for all i the inequalities $\lambda_i \geq 0, b_i \geq g_i(x^*)$ hold. Thus $\lambda_i(b_i - g_i(x^*)) \geq 0$ which implies

$$v(P) = f(x^*) \leq f(x^*) + \sum_{i=1}^{m} \lambda_i(b_i - g_i(x^*)) \leq v(L(\lambda))$$

$\square$

Theorem

Let x_L be the optimal solution of the Lagrange relaxation. If

$$g(x_L) \leq b$$

and

$$\sum_{i=1}^{m} \lambda_i (b_i - g_i(x_L)) = 0$$

then x_L is an optimal solution of (P)

Proof.

$g(x_L) \leq b$ means that x_L is a feasible solution of (P). For any feasible solution x of (P) it follows from the optimality of x_L that

$$f(x) \leq f(x) + \sum_{i=1}^{m} \lambda_i (b_i - g_i(x)) \leq f(x_L) + \sum_{i=1}^{m} \lambda_i (b_i - g_i(x_L)) = f(x_L),$$

i.e. x_L is at least as good as x. □

Example

$$\begin{aligned}\text{Max} \quad & 23x_1 + 19x_2 + 28x_3 + 14x_4 + 44x_5 \\ \text{s.t.} \quad & 5x_1 + x_2 + 6x_3 + 3x_4 + 5x_5 \leq 14 \\ & 2x_1 + x_2 - 3x_3 + 5x_4 + 6x_5 \leq 4 \\ & x_1 + 5x_2 + 8x_3 - 2x_4 + 8x_5 \leq 7 \\ & x_j \in \{0, 1\}\end{aligned}$$

Let $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 3$. Then the objective function of the Lagrange relaxation is:

$$\begin{aligned}&(23x_1 + 19x_2 + 28x_3 + 14x_4 + 44x_5) + (14 - 5x_1 - x_2 - 6x_3 - 3x_4 - 5x_5) \\ &+3(4 - 2x_1 - x_2 + 3x_3 - 5x_4 - 6x_5) + 3(7 - x_1 - 5x_2 - 8x_3 + 2x_4 - 8x_5) \\ &= 47 + 9x_1 + 0x_2 + 7x_3 + 2x_4 - 3x_5\end{aligned}$$

The only constraint is that all variables are binary. It implies that if a coefficient is positive in the objective function then the variable should be 1 in the optimal solution of the Lagrange problem, and if the coefficient is negative then the variable must be 0. As the coefficient of x_2 is zero, there are two optimal solutions: $(1, 0, 1, 1, 0)$ and $(1, 1, 1, 1, 0)$. The first one satisfies the optimality condition thus it is an optimal solution. The second one is infeasible.

Definition

Let f, h be two functions mapping from the n-dimensional Euclidean space into the real numbers. Further on Let Ω, Φ be two subsets of the n-dimensional Euclidean space. The problem

$$\max\{h(x) | x \in \Omega\} \qquad (P')$$

is a relaxation of the problem

$$\max\{f(x) | x \in \Phi\} \qquad (P)$$

if
(i) $\Phi \subseteq \Omega$ and
(ii) it is known as a priori, i.e. without solving the problems that $v(P') \geq v(P)$.

The general form of the B&B method

B&B divides the problem into subproblems and tries to fathom each subproblem by the help of a relaxation. A subproblem is fathomed in one of the following cases:

- The optimal solution of the relaxed problem satisfies the constraints of the unrelaxed problem and its relaxed and non-relaxed objective function values are equal.
- The infeasibility of the relaxed subproblem implies that the unrelaxed subproblem is infeasible as well.
- The upper bound provided by the relaxed subproblem is less or equal than the objective function value of the best known feasible solution.

The algorithm can stop if all subsets are fathomed.