Math 1010 Week 9

L'Hôpital's Rule, Taylor Series

Theorem 9.1 (Cauchy's Mean Value Theorem)**.** *If $f, g : [a, b] \longrightarrow \mathbb{R}$ are functions which are continuous on $[a, b]$ and differentiable on (a, b), and $g(a) \neq g(b)$, then there exists $c \in (a, b)$ such that:*

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Proof. **Exercise.** Apply Rolle's Theorem to:

$$h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a))$$

□

IMAGE

Theorem 9.2 (L'Hôpital's Rule)**.** *Let $c \in \mathbb{R}$. Let $I = (a, b)$ be an open interval containing c. Let f, g be functions which are differentiable at every point in $(a, c) \cup (c, b)$. Suppose:*

- $\lim_{x \to c} f(x)$ *and* $\lim_{x \to c} g(x)$ *are both equal to* 0 *or both equal to* $\pm\infty$.
- $\lim_{x \to c} \dfrac{f'(x)}{g'(x)}$ *exists.*

(• $g'(x) \neq 0$ for all $x \neq c$ in I.)

Then,

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}\ .$$

Proof. (Sketch) We consider the special case where:

- $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0.$
- f and g are continuous at $x = c$.

For such functions f, g, we have:

$$f(c) = g(c) = 0.$$

Hence:

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(c)}{g(x) - g(c)} = \frac{f'(t_x)}{g'(t_x)}$$

for some t_x between c and x by Cauchy's Mean Value Theorem.

As x approaches c, the element t_x lying between x and c must also approach c.

Hence, if the limit $\lim_{x\to c} \dfrac{f'(x)}{g'(x)}$ exists, then intuitively it follows that:

$$\begin{aligned}\lim_{x\to c} \frac{f(x)}{g(x)} &= \lim_{t_x\to c} \frac{f'(t_x)}{g'(t_x)} \\ &= \lim_{t\to c} \frac{f'(t)}{g'(t)}.\end{aligned}$$

(**Optional Exercise** : To prove the above equality rigorously, one could, for example, apply the sequential criterion for the limit of a function .) $\square$

9.1 Indeterminate Forms

- $\frac{0}{0}$

- $\frac{\infty}{\infty}$

- $0 \cdot \infty$

- $\infty - \infty$

- 0^0

- ∞^0

- 1^∞

Here, for example, 1^∞ represents the situation $\lim\limits_{x\to c} f(x)^{g(x)}$ where $\lim\limits_{x\to c} f(x) = 1$ and $\lim\limits_{x\to c} g(x) = \infty$.

Hence, the following limit corresponds to the indeterminate form 1^∞:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x.$$

Example 9.3. *Use l'Hôpital's rule to evaluate the following limits:*

1. $\lim\limits_{x\to 0}\dfrac{e^x - 1 - x - \frac{x^2}{2}}{x^3}$

Solution.

$$\lim_{x\to 0}\left(e^x - 1 - x - \frac{x^2}{2}\right) = 0.$$
$$\lim_{x\to 0} x^3 = 0.$$

$$\lim_{x\to 0}\frac{\left(e^x-1-x-\frac{x^2}{2}\right)'}{\left(x^3\right)'}=\lim_{x\to 0}\frac{e^x-1-x}{3x^2}\qquad\left(\to\frac{0}{0}\right)$$

$$\begin{aligned}\lim_{x\to 0}\frac{\left(e^x-1-x\right)'}{\left(3x^2\right)'}&=\lim_{x\to 0}\frac{e^x-1}{6x}\\&=\frac{1}{6}\lim_{x\to 0}\frac{e^x-1}{x}\\&=\frac{1}{6}\end{aligned}$$

Hence, by l'Hôpital's rule,

$$\begin{aligned}\lim_{x\to 0}\frac{e^x-1-x-\frac{x^2}{2}}{x^3}&=\lim_{x\to 0}\frac{\left(e^x-1-x-\frac{x^2}{2}\right)'}{\left(x^3\right)'}\\&=\lim_{x\to 0}\frac{\left(e^x-1-x\right)'}{\left(3x^2\right)'}\\&=\frac{1}{6}\end{aligned}$$

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2. $\lim\limits_{x\to 0^+} x^{\frac{1}{1+\ln x}}$

Solution. *(This limit corresponds to the indeterminate form 0^0.)*

For $x>0$, we have $x=e^{\ln x}$. Hence,

$$\lim_{x\to 0^+} x^{\frac{1}{1+\ln x}}=\lim_{x\to 0^+} e^{\left(\frac{1}{1+\ln x}\right)\ln x}=e^{\lim_{x\to 0^+}\frac{\ln x}{1+\ln x}}$$

The last equality holds because $f(x)=e^x$ is a continuous function.

The limit $\lim\limits_{x\to 0^+}\dfrac{\ln x}{1+\ln x}$ corresponds to the indeterminate form $\frac{-\infty}{-\infty}$. So, it is possible in this case to apply l'Hopital's rule to help find the limit.

$$\begin{aligned}\lim_{x\to 0^+}\frac{(\ln x)'}{(1+\ln x)'}&=\lim_{x\to 0^+}\frac{\frac{1}{x}}{\frac{1}{x}}\\&=\lim_{x\to 0^+}\frac{x}{x}\\&=1\end{aligned}$$

By l'Hopital's rule, it now follows that:

$$\lim_{x \to 0^+} \frac{\ln x}{1 + \ln x} = 1.$$

Hence,

$$\begin{aligned}
\lim_{x \to 0^+} x^{\frac{1}{1+\ln x}} &= e^{\lim_{x \to 0^+} \frac{\ln x}{1+\ln x}} \\
&= e^1 \\
&= e.
\end{aligned}$$

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3. $\displaystyle \lim_{x \to +\infty} x \left(\frac{\pi}{2} - \tan^{-1} x \right)$

Solution. *(This limit corresponds to the indeterminate form $\infty \cdot 0$.) Rewrite the limit as follows:*

$$\lim_{x \to +\infty} x \left(\frac{\pi}{2} - \tan^{-1} x \right) = \lim_{x \to +\infty} \frac{\frac{\pi}{2} - \tan^{-1} x}{\frac{1}{x}} \qquad \left(\to \frac{0}{0} \right).$$

Now, compute:

$$\begin{aligned}
\lim_{x \to +\infty} \frac{\left(\frac{\pi}{2} - \tan^{-1} x \right)'}{\left(\frac{1}{x} \right)'} &= \lim_{x \to +\infty} \frac{-\frac{1}{1+x^2}}{-\frac{1}{x^2}} \\
&= \lim_{x \to +\infty} \frac{x^2}{1 + x^2} \\
&= \lim_{x \to +\infty} \frac{1}{\left(1 + \frac{1}{x^2}\right)} \\
&= 1
\end{aligned}$$

Hence, by l'Hopital's rule,

$$\lim_{x \to +\infty} x \left(\frac{\pi}{2} - \tan^{-1} x \right) = 1.$$

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4. $\lim\limits_{x\to+\infty}(e^x+x)^{\frac{1}{x}}$

Solution. *(This limit corresponds to the indeterminate form ∞^0.) We have:*

$$\begin{aligned}\lim_{x\to+\infty}(e^x+x)^{\frac{1}{x}} &= \lim_{x\to+\infty}\left(e^{\ln(e^x+x)}\right)^{\frac{1}{x}}\\ &= \lim_{x\to+\infty} e^{\frac{\ln(e^x+x)}{x}}\\ &= e^{\lim\limits_{x\to+\infty}\frac{\ln(e^x+x)}{x}}\end{aligned}$$

The limit $\lim\limits_{x\to+\infty}\frac{\ln(e^x+x)}{x}$ *corresponds to the indeterminate form* $\frac{\infty}{\infty}$.

$$\begin{aligned}\lim_{x\to+\infty}\frac{(\ln(e^x+x))'}{(x)'} &= \lim_{x\to+\infty}\frac{e^x+1}{e^x+x}\\ &= \lim_{x\to+\infty}\frac{e^x\left(1+\frac{1}{e^x}\right)}{e^x\left(1+\frac{x}{e^x}\right)}\\ &= \lim_{x\to+\infty}\frac{1+\frac{1}{e^x}}{1+\frac{x}{e^x}}\\ &= 1.\end{aligned}$$

Hence, by l'Hopital's rule,

$$\begin{aligned}\lim_{x\to+\infty}\frac{\ln(e^x+x)}{x} &= \lim_{x\to+\infty}\frac{(\ln(e^x+x))'}{(x)'}\\ &= \lim_{x\to+\infty}\frac{e^x+1}{e^x+x}\\ &= 1.\end{aligned}$$

It now follows that:

$$\lim_{x\to+\infty}(e^x+x)^{\frac{1}{x}} = e^{\lim\limits_{x\to+\infty}\frac{\ln(e^x+x)}{x}} = e^1 = e.$$

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5. $\lim_{x\to 0} \frac{1 - x\cot x}{x \sin x}$

Solution. *(This limit corresponds to the indeterminate form $\frac{0}{0}$.) Note that* $\cot x = \frac{\cos x}{\sin x}$*. Rewrite the limit as follows:*

$$\lim_{x\to 0} \frac{1 - x\cot x}{x\sin x} = \lim_{x\to 0} \frac{\sin x - x\cos x}{x\sin^2 x} \qquad \left(\to \frac{0}{0}\right)$$

One's first instinct might be to differentiate both numerator and denominator right away. But this seems unwise, since, looking further down the road, we would have to deal with an indeterminate form whose denominator is $(x\sin^2 x)' = 2x\sin x\cos x + \sin^2 x$*. Repeating the differentiation of the numerator and denominator would only make the expression more and more complicated.*

A cleverer way would be to rewrite the limit as follows:

$$\lim_{x\to 0} \frac{\sin x - x\cos x}{x\sin^2 x} = \lim_{x\to 0} \frac{\sin x - x\cos x}{x^3} \cdot \frac{x^2}{\sin^2 x}.$$

This is motivated by the observation that $\sin^2 x$ *is very close to* x^2 *when* x *is close to* 0*.*

First, we have:

$$\lim_{x\to 0} \frac{x^2}{\sin^2 x} = \left(\lim_{x\to 0} \frac{x}{\sin x}\right)^2 = 1.$$

The limit $\lim_{x\to 0} \frac{\sin x - x\cos x}{x^3}$ *corresponds to the indeterminate form $\frac{0}{0}$. Differentiating both numerator and denominator, we have:*

$$\begin{aligned}\lim_{x\to 0} \frac{(\sin x - x\cos x)'}{(x^3)'} &= \lim_{x\to 0} \frac{\cos x + x\sin x - \cos x}{3x^2} \\ &= \lim_{x\to 0} \frac{x\sin x}{3x^2} \\ &= \lim_{x\to 0} \frac{\sin x}{3x} \\ &= \frac{1}{3}.\end{aligned}$$

Hence, by l'Hopital's rule we have:

$$\lim_{x\to 0} \frac{\sin x - x\cos x}{x^3} = \lim_{x\to 0} \frac{(\sin x - x\cos x)'}{(x^3)'} = \frac{1}{3}.$$

It now follows that:

$$
\begin{aligned}
\lim_{x\to 0} \frac{1 - x\cot x}{x\sin x} &= \lim_{x\to 0} \frac{\sin x - x\cos x}{x^3} \cdot \lim_{x\to 0} \frac{x^2}{\sin^2 x} \\
&= \frac{1}{3} \cdot 1 \\
&= \frac{1}{3}.
\end{aligned}
$$

Exercise 9.4. *1.* **WeBWorK**

2. **WeBWorK**

3. **WeBWorK**

4. **WeBWorK**

5. **WeBWorK**

6. **WeBWorK**

7. **WeBWorK**

8. **WeBWorK**

9. **WeBWorK**

Important Note. If $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ or $\pm\infty$, and $\displaystyle\lim_{x\to c} \frac{f'(x)}{g'(x)}$ does not exist, it **DOES NOT** follow that $\displaystyle\lim_{x\to c} \frac{f(x)}{g(x)}$ does not exist.

Example 9.5.

$$\lim_{x\to\infty} \frac{x + \sin x}{x}$$

Definition 9.6. *Given a function f which is n times differentiable at a. The* **n-th Taylor polynomial of f (centered) at** *a is:*

$$P(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Observe that:

$$P^{(k)}(a) = f^{(k)}(a),$$

for $k = 0, 1, 2, \ldots, n$.

Example 9.7. *The Taylor polynomials at $a = 0$ for various functions f are as follows:*

$f(x)$	$P(x)$
$\cos x$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!}$
$\sin x$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$
e^x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$
$\ln(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1} \frac{x^n}{n}$
$\frac{1}{1-x}$	$1 + x + x^2 + x^3 + \cdots + x^n$

Note, for example, that the 5-th and 6-th Taylor polynomials of $f(x) = \sin x$ at $x = 0$ both have degree 5. Hence, an n-th Taylor polynomial does not necessarily have degree n.

- Taylor polynomials of $f(x) = \sin x$ centered at $a = 0$.
- Taylor polynomials of $f(x) = \sin x$ centered at $a = \pi/2$.

Theorem 9.8 (Taylor's Formula)**.** *Let n be a positive integer, and $a \in \mathbb{R}$. Let f be a function which is $n+1$ times differentiable on an open interval I containing a. Let:*

$$\begin{aligned} P_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \\ &= f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 \\ &\quad + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

be the n-th Taylor polynomial of f at a. Then, for any $x \in I$, we have:

$$f(x) = P_n(x) + R_n(x),$$

where the **remainder term** $R_n(x)$ *is equal to:*

$$\frac{f^{(n+1)}(c)}{(n+1)!}\,(x-a)^{n+1}$$

for some c *between* a *and* x.

Note that the special case $n = 0$ is equivalent to (Lagrange's) Mean Value Theorem.

Proof. Recall that $P_n^{(k)}(a) = f^{(k)}(a)$ for $k = 0, 1, 2, \ldots, n$. Moreover, observe that $P_n^{(k)} = 0$ for $k > n$, since P_n is a polynomial of degree at most n.

Let:

$$F(x) = f(x) - P_n(x), \quad G(x) = (x-a)^{n+1}.$$

Then, $F(a) = G(a) = 0$, and by Cauchy's Mean Value Theorem (Cauchy's Mean Value Theorem.), we have:

$$\begin{aligned}
\frac{f(x) - P_n(x)}{(x-a)^{n+1}} &= \frac{F(x) - F(a)}{G(x) - G(a)} \\
&= \frac{F'(x_1)}{G'(x_1)} \\
&= \frac{f'(x_1) - P_n'(x_1)}{(n+1)(x_1 - a)^n}
\end{aligned}$$

for some x_1 between a and x.

Now let:

$$\begin{aligned}
F_1(x) = F'(x) = f'(x) - P_n'(x), \\
G_1(x) = G'(x) = (n+1)(x-a)^n.
\end{aligned}$$

Repeating the same procedure carried out before, we have:

$$\frac{f'(x_1) - P_n'(x_1)}{(n+1)(x_1 - a)^n} = \frac{F_1'(x)}{G_1'(x)} = \frac{f^{(2)}(x_2) - P_n^{(2)}(x_2)}{(n+1)n(x_2 - a)^{n-1}}$$

for some x_2 between a and x_1. Repeating this process $n + 1$ times, we have:

$$
\begin{aligned}
\frac{f(x) - P_n(x)}{(x-a)^{n+1}} &= \frac{f'(x_1) - P_n'(x_1)}{(n+1)(x_1 - a)^n} \\
&= \frac{f^{(2)}(x_2) - P_n^{(2)}(x_2)}{(n+1)n(x_2 - a)^{n-1}} \\
&\vdots \\
&= \frac{f^{(n)}(x_n) - P_n^{(n)}(x_n)}{(n+1)n(n-1)\cdots 2(x_n - a)} \\
&= \frac{f^{(n+1)}(x_{n+1}) - 0}{(n+1)!}
\end{aligned}
$$

for some x_{n+1} between a and x. Letting $c = x_{n+1}$, the theorem follows. $\square$

Definition 9.9. *Given a function f which is infinitely differentiable at a (i.e. $f^{(k)}(a)$ is defined for $k = 0, 1, 2, 3, \ldots$). The* **Taylor series of f (centered) at a** *is the power series:*

$$
\begin{aligned}
T(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \\
&= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots
\end{aligned}
$$

In general, for any power series of the form $S(x) = \sum_{k=0}^{\infty} a_k(x-a)^k$, the value of S at any given $c \in \mathbb{R}$ is by definition the limit:

$$
S(c) := \lim_{n\to\infty} \sum_{k=0}^{n} a_k(c-a)^k.
$$

Note that this limit does not necessarily exist. If it does exist, we say that the power series S **converges** at $x = c$, otherwise we say that it **diverges** at $x = c$.

Example 9.10. *The Taylor series at $a = 0$ for various functions f are as follows:*

$f(x)$	$T(x)$
$\cos x$	$\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}$
$\sin x$	$\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$
e^x	$\sum_{k=0}^{\infty} \frac{x^k}{k!}$
$\ln(1+x)$	$\sum_{k=1}^{\infty}(-1)^{k+1} \frac{x^k}{k}$
$\frac{1}{1-x}$	$\sum_{k=0}^{\infty} x^k$

Theorem 9.11 (Binomial Series)**.** *For $\alpha \in \mathbb{R}$, $|x| < 1$,*

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \cdots ,$$

where:

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}$$

Example 9.12. *In particular, for $|x| < 1$, we have:*

$$\begin{aligned}\sqrt{1+x} &= (1+x)^{1/2} \\ &= 1 + \frac{1}{2}x + \frac{(1/2)(1/2-1)}{2!}x^2 + \frac{(1/2)(1/2-1)(1/2-2)}{3!}x^3 + \cdots\end{aligned}$$

Example 9.13. *The Taylor $T(x)$ series of $f(x) = e^x$ at $a = 0$ converges everywhere. Moreover, for each $x \in \mathbb{R}$, we do have:*

$$T(x) = \sum_{k=0}^{\infty} \frac{1}{k!}\, x^k = e^x.$$

Similarly, for all $x \in \mathbb{R}$, we have:

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\, x^{2k+1} = \sin x$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\, x^{2k} = \cos x$$

However,

The Taylor series of $f(x) = \ln(1+x)$ at $a = 0$ is:

$$T(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\, x^k,$$

which converges only for $x \in (-1, 1]$.

For such x we do have:

$$T(x) = f(x).$$

In particular, we have:

$$\ln 2 = \ln(1+1) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\, 1^k = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Remark. *There are functions whose Taylor series converge everywhere, but not to the functions themselves.*