Math 1010 Week 8

Curve Sketching

8.1 Absolute/Relative (Global/Local) Extrema

Consider a function $f : A \longrightarrow \mathbb{R}$.

Definition 8.1.
- *If there is an element $c \in A$ such that: $f(c) \leq f(x)$ for all $x \in A$, we say that $f(c)$ is the (global/absolute)* **minimum** *of f.*
- *If there is an element $d \in A$ such that: $f(d) \geq f(x)$ for all $x \in A$, we say that $f(d)$ is the (global/absolute)* **maximum** *of f.*

Definition 8.2.
- *If $f(c) \leq f(x)$ for all x in an open interval containing c, we say that f has a* **local/relative minimum** *at c.*
- *If $f(c) \geq f(x)$ for all x in an open interval containing c, we say that f has a* **local/relative maximum** *at c.*

IMAGE
By KSmrq - http://commons.wikimedia.org/wiki/File:Extrema_example.svg ,
GFDL 1.2 , Link

Theorem 8.3 (First Derivative Test)**.** *Let $f : A \longrightarrow \mathbb{R}$ be a continuous function. For $c \in A$, if there exists an open interval (a, b) containing c such that $f'(x) < 0$ (in particular it exists) for all $x \in (a, c)$, and $f'(x) > 0$ for all $x \in (c, b)$, then f has a local minimum at c.*

Similarly, if $f'(x) > 0$ for all $x \in (a, c)$ and $f'(x) < 0$ for all $x \in (c, b)$, then f has a local maximum at c.

Note: In the special case that the domain of f is an open interval (a, b), if $f'(x) > 0$ for all $x \in (a, c)$, and $f'(x) < 0$ for all $x \in (c, b)$, then f has an absolute maximum at c.

Similarly f has an absolute minimum at c if each of the above inequalities is reversed.

Example 8.4.
- *In Example 7.6, the function has a local maximum at $x = -5$, and a local minimum at $x = 1$.*
- *In Example 7.7, the function has only one local extremum, namely a local minimum at $x = -1$. In fact, $f(-1) = 0$ is the absolute minimum of f.*

Exercise 8.5. $f(x) = x^{\frac{1}{3}} - \frac{1}{3}x - \frac{2}{3}$ *for* $x > 0$. *Show that* $f(x) \leq 0$ *for all* $x > 0$. *Then, deduce that:*

$$u^{\frac{1}{3}}v^{\frac{2}{3}} \leq \frac{1}{3}u + \frac{2}{3}v$$

for u*,* $v > 0$.

8.2 WeBWorK

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Theorem 8.6 (Second Derivative Test)**.** *Let f be a function twice differentiable at $c \in \mathbb{R}$, such that $f'(c) = 0$. If:*

- $f''(c) > 0$*, then f has a local minimum at c.*
- $f''(c) < 0$*, then f has a local maximum at c.*

Proof. **Sketch of Proof.** Suppose $f''(c) > 0$, by the definition of $f''(c)$ as the derivative of f' at c, we have:

$$0 < f''(c) = \lim_{h \to 0} \frac{f'(c+h) - f'(c)}{h} = \lim_{h \to 0} \frac{f'(c+h)}{h} .$$

It follows from the above identity that $f'(c+h)$ is > 0 for sufficiently small positive h, and < 0 for sufficiently small negative h.

Hence there is an open interval (a,b) containing c such that f' is negative on (a,c) and positive on (c,b). So, f has a local minimum at c by the First Derivative Test.

The case $f''(c) < 0$ may be proved similarly. □

Example 8.7. *Consider the function $f(x) = x^3 + 6x^2 - 15x + 7$ in Example 7.6, we have:*

$$f''(x) = 6x + 12$$

The function f has a two stationary points $c = -5, 1$ where $f'(c) = 0$.

Since:

$$f''(-5) = -18, \quad f''(1) = 18,$$

by the Second Derivative Test $f(-5)$ is a local maximum, and $f(1)$ is a local minimum. (This corroborates the conclusions of the First Derivative Test applied to the same function, see Example 8.4.)

Example 8.8. *Consider $g(x) = x^4$. Then, $g'(x) = 4x^3$, which implies that $c = 0$ is the only point where $g'(c) = 0$.*

The second derivative of g is $g''(x) = 12x^2$. Hence, $g''(c) = g''(0) = 0$.

In this case, no conclusion can be drawn from the Second Derivative Test, regarding whether $g(0)$ is a local minimum, maximum, or neither.

However, one can still apply the First Derivative Test to conclude that $f(0) = 0$ is a local minimum.

8.3 Concavity

Let f be a twice differentiable function. If f'' is positive (resp. negative) on an open interval (a,b), then the graph of f over (a,b) is **concave up** (resp. **down**). This is due to the fact that f'' being positive (resp. negative) corresponds to f' being increasing (resp. decreasing).

IMAGE
By dino -
http://en.wikipedia.org/wiki/File:Animated_illustration_of_inflection_point.gif
Public Domain, Link

A point on the graph of f where the concavity changes is called an **inflection point**. It corresponds to a point in the domain of f where f'' changes sign.

Example 8.9. *Sketch the graph of:*

$$f(x) = \frac{x^2 + x - 2}{x^2}$$

by first finding the following information about f:

1. *Domain.*

$$\{x \in \mathbb{R} : x \neq 0\} = (-\infty, 0) \cup (0, \infty)$$

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2. *x-intercepts (if sufficiently easy to find), and y-intercept. $f(x) = 0$ if and only if $x \neq 0$ and $x^2 + x - 2 = (x-1)(x+2) = 0$. Hence, the x-intercepts are:*

$$x = 1, -2.$$

In general, the y-intercept of the graph of a function is the value of the function at $x = 0$. In this case, 0 is not in the domain of f, hence the graph of f has no y-intercept.

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3. *Asymptotes (Horizontal, Vertical, Oblique)*

$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 1.$$

Hence, the graph of f has one horizontal asymptote: $y = 1$. The value $f(x)$ is defined for all $x \neq 0$. Hence, f, being a rational function, is continuous at all $x \neq 0$. So, there are no vertical asymptotes at $x \neq 0$. Near $x = 0$, we have:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = -\infty.$$

Hence, the graph of f has a vertical asymptote at $x = 0$. Since $f(x)$ approaches 1 as x approaches $\pm\infty$, the graph of f has no oblique asymptote.

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4. *Intervals where f is increasing/decreasing.*

$$\begin{aligned} f'(x) &= \frac{x^2(2x+1)-(x^2+x-2)2x}{(x^2)^2} \\ &= \frac{x^2(2x+1)-(x^2+x-2)2x}{(x^2)^2} \\ &= \frac{1}{x^3}(4-x) \end{aligned}$$

Hence, the points c where f' possibly changes sign are $c = 0, 4$.

$y = f(x)$:	$\searrow$	$\nearrow$	$\searrow$
$f'(x)$:	$-$	$+$	$-$
x:	$(-\infty, 0)$	$(0, 4)$	$(4, \infty)$

It follows from the sign chart that f is increasing on $(0, 4]$, and decreasing on $(-\infty, 0)$ and $[4, \infty)$.

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5. *"Turning Points" on the graph of f (i.e. points corresponding to local extrema).*

   It follows from the sign chart above that f as a local maximum at $x = 4$. The corresponding point on the graph is $(4, f(4)) = (4, 9/8)$.

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6. *Intervals where f is concave up/down.*

   The second derivative of f is:

$$f''(x) = \frac{1}{x^4}(2x - 12)$$

The points where f'' possibly changes sign are points p where $f''(p) = 0$, or where $f''(p)$ is undefined. In this case, there are two such point: $p = 0, 6$.

$y = f(x)$:	$\cap$	$\cap$	$\cup$
$f''(x)$:	$-$	$-$	$+$
x:	$(-\infty, 0)$	$(0, 6)$	$(6, \infty)$

It follows from this sign chart that f is concave up on $(6, \infty)$, and concave down on $(-\infty, 0)$ and $(0, 6)$.

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7. *Inflection points on the graph of f. It follows from the sign chart for f'' that $(6, f(6)) = (6, 10/9)$ is the only reflection point on the graph of f.*

Graph

$$y = \frac{x^2 + x - 2}{x^2}$$

Example 8.10. *Following the guidelines of the previous example, sketch the graph of:*

1. $f(x) = |x + 1|\,(3 - x)$
2. $f(x) = x + \frac{1}{|x|}$

Solution. 1. *Domain:* $\mathbb{R}$. *x-intercepts:* $x = -1, 3$. *y-intercept:* $y = f(0) = 3$. *Asymptotes: None.*

$$f'(x) = \begin{cases} 2x - 2 & x < -1; \\ \textit{undefined} & x = -1; \\ -2x + 2 & x > -1. \end{cases}$$

Critical points: $c = -1, 1$.

$$f''(x) = \begin{cases} 2 & x < -1; \\ \textit{undefined} & x = -1; \\ -2 & x > -1. \end{cases}$$

Inflection point: $p = -1$.

$y = f(x)$:	$\cup \searrow$	$\cap \nearrow$	$\cap \searrow$
$f'(x)$:	$-$	$+$	$-$
$f''(x)$:	$+$	$-$	$-$
x:	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$

2. *In general, if one can rewrite a function f (e.g. using long division if f is a rational function) in the form:*

$$f(x) = mx + b + g(x),$$

such that $\lim_{x \to \pm\infty} g(x) = 0$, and m, b are constants, then one can readily conclude that $y = mx + b$ is an asymptote for the graph of f. If $m \neq 0$, we call $y = mx + b$ an oblique asymptote. If $m = 0$, then $y = b$ is a horizontal asymptote. In this example, since $\lim_{x \to \pm\infty} \frac{1}{|x|} = 0$, the graph of $f(x) = x + \frac{1}{|x|}$ has an oblique asymptote: $y = x$. We leave the rest of the calculations as an exercise. Hint :

$$f(x) = \begin{cases} x - \frac{1}{x} & x < 0; \\ x + \frac{1}{x} & x > 0. \end{cases}$$

The resulting graph is as follows:

Example 8.11. *Consider the function $f(x) = \frac{5x^2+2}{x+1}$. We have:*

$$f(x) = 5x - 5 + \frac{7}{x+1}$$

Since $\lim_{x \to \pm\infty} \frac{7}{x+1} = 0$, the graph of f approaches the line $y = 5x - 5$ as x approaches $\pm\infty$.

We conclude that $y = 5x - 5$ is an oblique asymptote for the graph of f.

In general, if the graph of f approaches the line correpsonding to $l(x) = mx + b$, as x tends to $\pm\infty$, we have:

$$m = \lim_{x \to \pm\infty} \frac{f(x)}{x}.$$

and

$$b = \lim_{x \to \pm\infty} (f(x) - mx).$$

Example 8.12. *Let* $g(x) = \sqrt{x^2+1} + 4$.

Exercise.

$$\lim_{x\to\infty} \frac{f(x)}{x} = 1.$$

This suggests that $y = f(x)$ *approaches* $y = x + b$ *as* $x \to \infty$*, where:*

Exercise.

$$b = \lim_{x\to\pm\infty} (f(x) - 1 \cdot x) = 4.$$

Hence, $y = f(x)$ *approaches* $y = x + 4$ *as* x *tends to* ∞.

Similary, as x *tends to* $-\infty$*, we have:*

Exercise.

$$m = \lim_{x\to-\infty} \frac{f(x)}{x} = -1.$$

$$b = \lim_{x\to\pm-\infty} (f(x) - 1 \cdot x) = 4.$$

So, $y = f(x)$ *approaches* $y = -x + 4$ *as* x *tends to* $-\infty$.

We conclude that the graph of f *has two oblique asymptotes:*

$$y = x + 4,$$
$$y = -x + 4.$$