Math 1010 Week 7

Mean Value Theorem

Theorem 7.1 (Extreme Value Theorem)**.** *If f is a* <u>*continuous*</u> *function defined on a* <u>*closed*</u> *interval $[a, b]$, then it attains both a maximum value and a minimum value on $[a, b]$.*

7.1 The Mean Value Theorem

Theorem 7.2 (Rolle's Theorem)**.** *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function which is continuous on $[a, b]$ and differentiable on (a, b) (i.e. $f'(x)$ exists for all $x \in (a, b)$). If $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.*

IMAGE

Proof. **Sketch of Proof.** First, it follows from the Extreme Value Theorem that f has an absolute maximum or minimum at a point c in (a, b). It may then be shown that:

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = 0,$$

using that fact that if $f(c)$ is an absolute extremum, then $\dfrac{f(c+h) - f(c)}{h}$ is both ≤ 0 and ≥ 0. □

Theorem 7.3 (Mean Value Theorem MVT)**.** *(Also known as* **Lagrange's Mean Value Theorem***)*

If a function $f : [a, b] \longrightarrow \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

IMAGE

Proof. Let f be a function which satisfies the conditions of the theorem. Define a function $g : [a, b] \longrightarrow \mathbb{R}$ as follows:

$$g(x) = f(x) - \left[\left(\frac{f(b) - f(a)}{b - a} \right) (x - a) + f(a) \right], \quad x \in [a, b].$$

(Intuitively, g is obtained from f by subtracting from f the line segment joining $(a, f(a))$ and $(b, f(b))$.) Observe that:

$$g(a) = g(b) = 0,$$

so the function g satisfies the conditions of Rolle's Theorem. Hence, there exists $c \in (a, b)$ such that:

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a},$$

which implies that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. □

7.2 WeBWorK

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7.3 Applications of the Mean Value Theorem

Theorem 7.4. *Let f be a differentiable function on an open interval (a, b). If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on (a, b).*

Proof. **Exercise.** For any $x_1, x_2 \in (a, b)$, show that the difference $f(x_2) - f(x_1)$ is equal to 0. □

Theorem 7.5. *Let f be a differentiable function on an open interval (a, b). If $f'(x) > 0$ (resp. $f'(x) < 0$) for all $x \in (a, b)$, then f is* **strictly increasing** *(resp.* **strictly decreasing***) on (a, b).*

Remark: *If f is moreover continuous on $[a, b]$, then f is increasing (resp. decreasing) on $[a, b]$ if f' is positive (resp. negative) on (a, b).*

Proof. We will prove the case $f'(x) > 0$.

Suppose $f'(x) > 0$ for all $x \in (a, b)$. Given any $x_1, x_2 \in (a, b)$, such that $x_1 < x_2$, by the MVT there exists $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} .$$

By the condition $f'(x) > 0$ for all $x \in (a, b)$, we have $f'(c) > 0$. Also, $x_2 - x_1 > 0$. Hence:

$$f(x_2) - f(x_1) = f'(c) \cdot (x_2 - x_1) > 0.$$

This shows that f is increasing on (a, b). □

Example 7.6. *Find the intervals where the function $f(x) = x^3 + 6x^2 - 15x + 7$ is increasing/decreasing.*

Solution. *We apply Theorem 7.5.*

First, we find:

$$f'(x) = 3x^2 + 12x - 15$$

Observe that f' is defined and continuous everywhere. Hence, the intervals where f' is positive/negative are separated by points c where $f'(c) = 0$. (Such points are called **stationary points** *of f). Setting:*

$$f'(c) = 3c^2 + 12c - 15 = 3(c^2 + 4c - 5) = 3(c + 5)(c - 1) = 0,$$

we see that the points where f' possibly changes sign are:

$$c = -5, 1$$

Consider now the **sign chart***:*

f:	$\nearrow$		$\searrow$		$\nearrow$
$f'(x)$:	$+$	0	$-$	0	$+$
x:	$(-\infty, -5)$	-5	$(-5, 1)$	1	$(1, \infty)$

It now follows from Theorem 7.5 and the continuity of f that:

- *f is increasing on the intervals $(-\infty, -5]$ and $[1, \infty)$.*
- *f is decreasing on the interval $[-5, 1]$.*

Example 7.7. *Let:*

$$f(x) = \begin{cases} (x+1)^2, & x < 0; \\ x + 1, & x \geq 0. \end{cases}$$

Find the intervals where the function f is increasing/decreasing.

Solution. *We carry out the same steps as in the previous example. We leave it as an exercise to show that:*

$$f'(x) = \begin{cases} 2x + 2, & x < 0; \\ \textit{undefined}, & x = 0; \\ 1, & x > 0. \end{cases}$$

Note that f' is not defined everywhere. In this case, the points where f' possibly changes sign are points c where:

$$f'(c) = 0 \quad \textit{or} \quad f'(c) \textit{ is undefined.}$$

Such points are called the **critical points** *of f. (Note that the set of stationary points is a subset of critical points). Constructing a sign chart as in the previous example, we have:*

f:	$\searrow$		$\nearrow$		$\nearrow$
$f'(x)$:	$-$	0	$+$	*undefined*	$+$
x:	$(-\infty, -1)$	-1	$(-1, 0)$	0	$(0, \infty)$

Hence, by Theorem 7.5, f is <u>*decreasing*</u> *on:*

$$(-\infty, -1],$$

and increasing on both $[-1, 0]$ and $[0, \infty)$. Since f is continuous at $x = 0$, we conclude that f is <u>*increasing*</u> *on:*

$$[-1, \infty).$$

Exercise 7.8. *Use the mean value theorem to prove that for $x > 0$,*

$$\frac{x}{1+x} < \ln(1+x) < x.$$

Hence, deduce that for $x > 0$,

$$\frac{1}{1+x} < \ln\left(1 + \frac{1}{x}\right) < \frac{1}{x}.$$

Solution. *We first show that:*

$$\ln(1+x) < x \ .$$

Consider the function:

$$f(x) = \ln(1+x) - x \ .$$

Then, $f(0) = 0$*, and* $f'(x) = \dfrac{-x}{1+x}$*.*

Hence, $f'(x) < 0$ *for all* $x > 0$*.*

For any $x > 0$*, by the Mean Value Theorem we have:*

$$\frac{f(x) - f(0)}{x - 0} = f'(c)$$

for some $c \in (0, x)$*. Since* $c > 0$*, we have* $f'(c) < 0$*, which implies that:*

$$\frac{f(x) - f(0)}{x - 0} < 0.$$

Since $x > 0$*, we conclude that* $\ln(1+x) - x = f(x) = f(x) - f(0) < 0$*. We conclude that:*

$$\ln(1+x) < x,$$

for all $x > 0$*.*

To show that $\frac{x}{1+x} < \ln(1+x)$*, we proceed similarly.*

Consider:

$$g(x) = \ln(1+x) - \frac{x}{1+x}.$$

Then, $g(0) = 0$*, and:*

$$\begin{aligned} g'(x) &= \frac{1}{1+x} - \frac{(1+x)1 - x(1)}{(1+x)^2} \\ &= \frac{x}{(1+x)^2} \\ &> 0 \end{aligned}$$

for all $x > 0$*.*

Hence, for all $x > 0$*, by the Mean Value Theorem we have:*

$$\frac{g(x) - g(0)}{x - 0} = g'(c) > 0,$$

where c *is some element which lies in* $(0, x)$*.*

This shows that $\ln(1+x) - \frac{x}{1+x} = g(x) > 0$*. Hence,*

$$\ln(1+x) > \frac{x}{1+x}$$

for $x > 0$.

Finally, for all $t > 0$*, we have* $\frac{1}{t} > 0$*. Applying the inequality:*

$$\frac{x}{1+x} < \ln(1+x) < x$$

to $x = \frac{1}{t}$*, we have:*

$$\frac{1/t}{1+1/t} < \ln\left(1+\frac{1}{t}\right) < \frac{1}{t},$$

which is equivalent to:

$$\frac{1}{t+1} < \ln\left(1+\frac{1}{t}\right) < \frac{1}{t}.$$