Math 1010 Week 2

Functions

2.1 Sandwich Theorem - Continued

Example 2.1. *1. Find the following limit:* $\lim_{n\to\infty} \frac{\sin(2^n) + (-1)^n \cos(2^n)}{n^3}$.

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2. - *Prove that* $\frac{2^n}{n!} \leq \frac{4}{n}$ *for all natural numbers* $n \geq 2$.
   - *Then, show that* $\lim_{n\to\infty} \frac{2^n}{n!} = 0$.

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3. *Suppose* $0 < a < 1$. *Let* $b = \frac{1}{a} - 1$. *For* $n \geq 2$, *use the binomial theorem to show that*

$$\frac{1}{a^n} \geq \frac{n(n-1)}{2} b^2.$$

Then, show that:

$$\lim_{n\to\infty} na^n = 0.$$

Exercise 2.2. *Using the inequality:*

$$\frac{1}{\sqrt{n^2+n}} \leq \frac{1}{\sqrt{n^2+r}} \leq \frac{1}{\sqrt{n^2+1}}, \quad \textit{for } r = 1, 2, 3, \cdots, n,$$

prove that:

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}\right)=1.$$

Solution. *We have:*

$$\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}\leq\underbrace{\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+1}}+\cdots+\frac{1}{\sqrt{n^2+1}}}_{n\text{ times}}=\frac{n}{\sqrt{n^2+1}},$$

and:

$$\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}\geq\underbrace{\frac{1}{\sqrt{n^2+n}}+\frac{1}{\sqrt{n^2+n}}+\cdots+\frac{1}{\sqrt{n^2+n}}}_{n\text{ times}}=\frac{n}{\sqrt{n^2+n}}.$$

Since:

$$\lim_{n\to\infty}\frac{n}{\sqrt{n^2+1}}=\lim_{n\to\infty}\frac{n}{n\sqrt{1+\frac{1}{n^2}}}=\lim_{n\to\infty}\frac{1}{1+\frac{1}{n^2}}=1,$$

and:

$$\lim_{n\to\infty}\frac{n}{\sqrt{n^2+n}}=\lim_{n\to\infty}\frac{n}{n\sqrt{1+\frac{1}{n}}}=\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}=1,$$

by the Sandwich Theorem we conclude that:

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}\right)=1.$$

2.2 Functions

Definition 2.3. *A function:*

$$f:A\longrightarrow B$$

is a rule of correspondence from one set A (called the **domain***) to another set B (called the* **codomain***).*

Under this rule of correspondence, each element $x\in A$ corresponds to exactly one element $f(x)\in B$, called the **value** *of f at x.*

In the context of this course, the domain A is usually some subset (intervals, union of intervals) of $\mathbb{R}$, while the codomain B is often presumed to be $\mathbb{R}$. Sometimes, the domain of a function is not explicitly given, and a function is simply defined by an expression in terms of an independent variable.

For example,

$$f(x) = \sqrt{\frac{x+1}{x-2}}$$

In this case, the domain of f is assumed to be the **natural domain** (or **maximal domain** , **domain of definition**), namely the largest subset of $\mathbb{R}$ on which the expression defining f is well-defined.

Example 2.4. *For the function:*

$$f(x) = \sqrt{\frac{x+1}{x-2}},$$

the natural domain is:

$$\begin{aligned}\text{Domain}(f) &= \left\{ x \in \mathbb{R} \;\middle|\; \frac{x+1}{x-2} \geq 0 \right\} \\ &= (-\infty, -1] \cup (2, \infty).\end{aligned}$$

2.2.1 Graphs of Functions

For $f : A \longrightarrow B$ where A, B are subsets of $\mathbb{R}$, it is often useful to consider the **graph** of f, namely the set of all points (x, y) in the xy-plane where $x \in A$ and $y = f(x)$. By definition, any function f takes on a unique value $f(x)$ for each x in its domain, hence the graph of f necessarily passes the so-called "**vertical line test**", namely, any vertical line which one draws in the xy-plane intersects the graph of f *at most once*.

The graph of a circle, for example, is not the graph of any function, since there are vertical lines which intersect the graph twice.

Exercise 2.5. *Graph the functions* $f(x) = \dfrac{x}{2}$ *and* $g(x) = \dfrac{4}{x} - 1$ *together, to identify values of* x *for which*

$$\frac{x}{2} > \frac{4}{x} - 1.$$

Confirm your answer by solving the inequality algebraically.

Solution. *The inequality holds if and only if:*

$$x \in (-4, 0) \cup (2, \infty)$$

2.2.2 Algebraic Operations on Functions

Definition 2.6. *Given two functions:*

$$f, g : A \longrightarrow \mathbb{R},$$

- *Their* **sum/difference** *is:*

$$f \pm g : A \longrightarrow \mathbb{R},$$
$$(f \pm g)(a) := f(a) \pm g(a), \quad \textit{for all} a \in A;$$

- *Their* **product** *is:*

$$fg : A \longrightarrow \mathbb{R},$$
$$fg(a) := f(a)g(a), \quad \textit{for all} a \in A;$$

- *The* **quotient function** $\frac{f}{g}$ *is:*

$$\frac{f}{g} : A' \longrightarrow \mathbb{R},$$
$$\frac{f}{g}(a) := \frac{f(a)}{g(a)}, \quad \textit{for all} a \in A',$$

  where

$$A' = \{a \in A : g(a) \neq 0\}.$$

More generally, For:

$$f : A \longrightarrow \mathbb{R},$$
$$g : B \longrightarrow \mathbb{R},$$

we define $f \pm g$ and fg as follows:

$$f \pm g : A \cap B \longrightarrow \mathbb{R},$$
$$f \pm g(x) := f(x) \pm g(x), \quad x \in A \cap B.$$

$$fg : A \cap B \longrightarrow \mathbb{R},$$
$$fg(x) := f(x)g(x), \quad x \in A \cap B.$$

Similary, we define:

$$\frac{f}{g} : A \cap B' \longrightarrow \mathbb{R},$$
$$\frac{f}{g}(x) = \frac{f(x)}{g(x)}, \quad x \in A \cap B',$$

where $B' = \{b \in B : g(b) \neq 0\}$.

2.2.3 Composition of Functions

Given two functions:

$$f : A \longrightarrow B, \quad g : B \longrightarrow C,$$

the **composite function** $g \circ f$ is defined as follows:

$$g \circ f : A \longrightarrow C,$$
$$(g \circ f)(a) := g(f(a)), \quad \text{for all} a \in A.$$

More generally, the domain of $g \circ f$ is defined to be:

$$\text{Domain}(g \circ f) = \{a \in \text{Domain}(f) : f(a) \in \text{Domain}(g)\}.$$

2.2.4 Inverse of a Function

The **range** or **image** of a function $f : A \longrightarrow B$ is the set of all $b \in B$ such that $b = f(a)$ for some $a \in A$.

Notation.

$$\text{Image}(f) = \text{Range}(f) := \{b \in B \; : \; b = f(a) \, \textit{for some} a \in A\}.$$

Note that the range of f is not necessarily equal to the codomain B.

Definition 2.7. *If* $\text{Range}(f) = B$*, we say that* f *is* **surjective** *or* **onto** .

Definition 2.8. *If $f(a) \neq f(a')$ for all $a, a' \in \mathrm{Domain}(f)$ such that $a \neq a'$, we say that f is* **injective** *or* **one-to-one** .

If $f : A \longrightarrow B$ is injective, then there exists an **inverse function**:

$$f^{-1} : \mathrm{Range}(f) \longrightarrow A$$

such that $f^{-1} \circ f$ is the **identity function** on A, and $f \circ f^{-1}$ is the identity function on $\mathrm{Range}(f)$, that is:

- $$f^{-1}(f(a)) = a, \quad \text{for all} a \in A,$$
- $$f(f^{-1}(b)) = b, \quad \text{for all} b \in \mathrm{Range}(f).$$

Example 2.9.

$$f : \mathbb{R} \longrightarrow \mathbb{R},$$
$$f(x) := x^2, \quad x \in \mathbb{R}.$$

is not injective, hence it has no inverse.

On the other hand,

$$f : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R},$$
$$f(x) := x^2, \quad x \in \mathbb{R}_{\geq 0};$$

is injective. It's range is $\mathrm{Range}(f) = \mathbb{R}_{\geq 0}$. Its inverse is:

$$f^{-1} : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$$
$$f^{-1}(y) = \sqrt{y}, \quad y \in \mathbb{R}_{\geq 0}.$$

Similarly,

$$g : \mathbb{R}_{\leq 0} \longrightarrow \mathbb{R},$$
$$g(x) := x^2, \quad x \in \mathbb{R}_{\leq 0};$$

is also injective, with $\mathrm{Range}(g) = \mathbb{R}_{\geq 0}$, and inverse:

$$g^{-1} : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\leq 0}$$
$$g^{-1}(y) = -\sqrt{y}, \quad y \in \mathbb{R}_{\geq 0}.$$

2.3 Piecewise Defined Functions

Example 2.10. •

$$f(x) = \begin{cases} -x+1 & \textit{if} -2 \leq x < 0 \\ 3x & \textit{if} 0 \leq x \leq 5 \end{cases}$$

- *The* **absolute value function**

$$|x| = \begin{cases} -x & \textit{if} x < 0 \\ x & \textit{if} x \geq 0 \end{cases}$$

Exercise 2.11. *Let* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *be the function defined by:*

$$f(x) = -3x + 4 - |x+1| - |x-1|$$

for any $x \in \mathbb{R}$.

1. *Express the 'explicit formula' of the function* f *as that of a piecewise defined function, with one 'piece' for each of* $(-\infty, -1)$, $[-1, 1)$, $[1, +\infty)$.
2. *Sketch the graph of the function* f.
3. *Is* f *an injective function on* $\mathbb{R}$*? Justify your answer.*
4. *What is the image of* $\mathbb{R}$ *under the function* f*?*

Solution.

1.
$$f(x) = \begin{cases} -x+4 & \textit{if} \quad x < -1 \\ -3x+2 & \textit{if} \quad -1 \leq x < 1 \\ -5x+4 & \textit{if} \quad x \geq 1 \end{cases}$$

2.

3. f *is strictly decreasing on* $\mathbb{R}$. *Hence,* f *is injective on* $\mathbb{R}$.

4. *The image of* $\mathbb{R}$ *under* f *is* $\mathbb{R}$.

2.4 WeBWorK

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2. **WeBWorK**
3. **WeBWorK**
4. **WeBWorK**
5. **WeBWorK**
6. **WeBWorK**

2.5 Even and Odd Functions

Definition 2.12. *Let f be a real-valued function defined on real numbers.*

- *It is said to be* **even** *if for any $x \in \mathrm{Domain}(f)$, $-x$ also lies in $\mathrm{Domain}(f)$ and:*
$$f(-x) = f(x).$$
- *It is said to be* **odd** *if for any $x \in \mathrm{Domain}(f)$, $-x$ also lies in $\mathrm{Domain}(f)$ and:*
$$f(-x) = -f(x).$$

Example 2.13.

1. *The polynomial $f(x) = x^4 + x^2 + 1$ is even, while the polynomial $g(x) = x^5 + x^3 + x$ is odd.*
2. *The function $f(x) = \cos x$ is even, while $f(x) = \sin x$ is odd.*
3. *The absolute value function is even.*

Fact 2.14.

1. *The sum of two even (resp. odd) functions is even (resp. odd).*
2. *The product of two even functions is even.*
3. *The product of two odd functions is also even.*
4. *The product of an even function with an odd function is odd.*
   For example, $f(x) = x\,|x|$ is odd.