Math 1010 Week 13

Definite Integrals

13.1 Motivation

Given a continuous function over a closed interval. We want to approximate the area of the region bounded by the graph of the function and the x-axis.

One way to do so is by viewing the region roughly as a union of sequence of rectangles, and then adding up the areas of these rectangles.

IMAGE
5 rectangles.

IMAGE
10 rectangles.

Intuitively, we see that the more (and smaller) rectangles are used, the more closely their union approximates the region in question.

IMAGE

Definition 13.1. *Let n be a positive integer.*

Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function on a closed interval.

Let:

$$\Delta x = \frac{b - a}{n}.$$

The **Left Riemann Sum** *of f over $[a, b]$ associated with n subintervals of equal lengths is:*

$$
\begin{aligned}
LS_n(f) = \sum_{k=0}^{n-1} f(a + k\Delta x)\Delta x \\
= \Delta x \Big[f(a) + f(a + \Delta x) + f(a + 2\Delta x) + \dots \\
\dots + f(a + (n-1)\Delta x) \Big]
\end{aligned}
$$

Each summand may be thought of as the area of the rectangle whose base is the subinterval $[a + k\Delta x, a + (k + 1)\Delta x]$, and whose height is the value of f at the left endpoint of the subinterval.

IMAGE$y = f(x) f(x)\Delta x x$

Definition 13.2. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function on a closed interval. The* **definite integral** $\int_a^b f(x)\, dx$ *of f over $[a, b]$ is equal to the limit as n tends to infinity of the left Riemann sum defined previously. That is:*

$$
\begin{aligned}
\int_a^b f(x)\, dx &= \lim_{n\to\infty} LS_n(f) \\
&= \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=0}^{n-1} f\left(a + \frac{k(b-a)}{n}\right)
\end{aligned}
$$

It is an established theorem that the limit exists if f is continuous.

(In fact: One could define the definite integral in terms of the Right Riemann Sum or the Midpoint Riemann Sum. All these sums tend to same limit in the case where f is continuous.) Our eventual goal is to show that if F is an antiderivative of a continuous function f, then:

$$
\int_a^b f(x)\, dx = F(x)\Big|_a^b := F(b) - F(a).
$$

- **Integration by Substitution**

$$
\int_a^b f(u(x))u'(x)\, dx = \int_{u(a)}^{u(b)} f(u)\, du = F(u(b)) - F(u(a))
$$

if F is an antiderivative of f.

- **Integration by Parts**

$$\int_a^b u(x)v'(x)dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)\,dx.$$

- **Integration by Trigonometric Substitution**

$$\int_{-3}^{3} \frac{dx}{\sqrt{3^2+x^2}} = \int_{-\pi/4}^{\pi/4} \cos\theta \sec^2\theta d\theta$$

- **Reduction Formulas**

$$\int_0^{\pi/2} \cos^n x\,dx = \left(\frac{1}{n}\cos^{n-1} x \sin x\right)\Big|_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x\,dx.$$

Before we prove the main theorem, we first state a couple of preliminary results.

Fact 13.3. *For a continuous function f on $[a, b]$, we have:*

$$\int_a^a f(x)\,dx = 0.$$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

Fact 13.4. *Let f be a continuous function on an interval I. For all $a, b, c \in I$, we have:*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$$

If f is an odd continuous function, then:

$$\begin{aligned}\int_{-a}^{a} f(x)dx &= \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx \\ &= \int_{-a}^{0} -(f(-x))dx + \int_{0}^{a} f(x)dx \\ &= \underbrace{\int_{t=a}^{t=0} (f(t))dt}_{t=-x} + \int_{0}^{a} f(x)dx \\ &= \int_{a}^{a} f(x)dx \\ &= 0\end{aligned}$$

If f is an even continuous function, then:

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

Claim 13.5. *Let f, g be continuous functions on $[a, b]$. If $f(x) \leq g(x)$ for all $x \in [a, b]$, then:*

$$\int_{a}^{b} f(x)\,dx \leq \int_{a}^{b} g(x)\,dx.$$

Example 13.6. *Find the area of the region in the xy-plane bounded between the graph of $y = x^2 - 2x - 3$ and the x-axis over the interval $[1, 5]$.*

IMAGE

The geometric area of the region described is equal to:

$$\int_{1}^{5} \left|x^2 - 2x - 3\right|\,dx$$

Consider the sign chart for the values of $f(x) = x^2 - 2x - 3 = (x + 1)(x - 3)$ over the interval $[1, 5]$:

$f(x)$:	$-$	0	$+$
x:	$[1, 3)$	3	$(3, 5]$

Hence,

$$\int_1^5 \left|x^2 - 2x - 3\right| \, dx$$

$$\begin{aligned}
&= \int_1^3 \left|x^2 - 2x - 3\right| \, dx + \int_3^5 \left|x^2 - 2x - 3\right| \, dx \\
&= \int_1^3 -\left(x^2 - 2x - 3\right) \, dx + \int_3^5 \left(x^2 - 2x - 3\right) \, dx \\
&= -\left(\frac{1}{3}x^3 - x^2 - 3x\right)\Bigg|_1^3 + \left(\frac{1}{3}x^3 - x^2 - 3x\right)\Bigg|_3^5 \\
&= \frac{16}{3} + \frac{32}{3} \\
&= 16
\end{aligned}$$

Theorem 13.7. *(***Mean Value Theorem for Integrals***) Let f be a continuous function on $[a, b]$. There exists $c \in [a, b]$ such that:*

$$f(c) = \frac{1}{b-a} \int_a^b f(x) \, dx.$$

Proof. Since f is continuous on $[a, b]$, by the Extreme Value Theorem it has a maximum value M and minimum value m on $[a, b]$.

In other words,

$$m \leq f(x) \leq M$$

for all $x \in [a, b]$. Hence:

$$\underbrace{\int_a^b m \, dx}_{m(b-a)} \leq \int_a^b f(x) \, dx \leq \underbrace{\int_a^b M \, dx}_{M(b-a)}.$$

Dividing each expression by $b - a$, we have:

$$m \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq M.$$

Let x_1, x_2 be elements in $[a, b]$ such that $M = f(x_1)$ and $m = f(x_2)$. Since f is continuous on $[a, b]$, and $\frac{1}{b-a} \int_a^b f(x) \, dx$ is a number between $f(x_1)$ and $f(x_2)$, by the Intermediate Value Theorem there exists c between x_1 and x_2 such that:

$$f(c) = \frac{1}{b-a} \int_a^b f(x) \, dx.$$

This c lies in $[a, b]$, since x_1, x_2 lies in $[a, b]$. □

Theorem 13.8 (Fundamental Theorem of Calculus Part I). *Let f be a continuous function on $[a, b]$. Define a function $F : [a, b] \longrightarrow \mathbb{R}$ as follows:*

$$F(x) = \int_a^x f(t)\,dt, \quad x \in [a, b].$$

Then, F is continuous on $[a, b]$ and differentiable on (a, b), with:

$$F'(x) = f(x)$$

for all $x \in (a, b)$. Equivalently:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Proof. By definition:

$$\begin{aligned} F'(x) &= \lim_{h\to 0}\frac{F(x+h) - F(x)}{h}. \\ &= \lim_{h\to 0}\frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}. \\ &= \lim_{h\to 0}\frac{\int_x^{x+h} f(t)\,dt}{h}. \end{aligned}$$

By the Mean Value Theorem for Integrals, there exists $c_h \in [x, x+h]$ such that:

$$f(c_h) = \frac{\int_x^{x+h} f(t)\,dt}{h}.$$

Hence:

$$F'(x) = \lim_{h\to 0} f(c_h) = f(x),$$

since for any h the number c_h lies between x and $x+h$, and f is continuous.

We leave the proof of the continuity of F on $[a, b]$ as an exercise. □

Corollary 13.9. *Let f be a continuous function. Let g and h be differentiable functions. Then:*

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))h'(x) - f(g(x))g'(x).$$

Example 13.10. *Evaluate:*

$$\frac{d}{dx}\int_{\sin x}^{x^3+1} e^{-t^2}\,dt$$

$$\begin{aligned}\frac{d}{dx}\int_{\sin x}^{x^3+1} e^{-t^2}\,dt &= e^{\left(-(x^3+1)^2\right)}(x^3+1)' - e^{\left(-(\sin x)^2\right)}(\sin x)' \\ &= e^{\left(-(x^3+1)^2\right)}\cdot 3x^2 - e^{\left(-(\sin x)^2\right)}\cos x\end{aligned}$$

Example 13.11. *Evaluate:*

$$\lim_{h\to 0^+}\frac{1}{\ln(1+h)}\int_2^{2+h}\sqrt{t^4+1}\,dt$$

We have:

$$\lim_{h\to 0^+}\frac{1}{\ln(1+h)}\int_2^{2+h}\sqrt{t^4+1}\,dt = \lim_{h\to 0^+}\frac{\int_2^{2+h}\sqrt{t^4+1}\,dt}{\ln(1+h)} \tag{13.1}$$

Computing the limits of the numerator and denominator separately, we have:

$$\lim_{h\to 0^+}\int_2^{2+h}\sqrt{t^4+1}\,dt = \int_2^2\sqrt{t^4+1}\,dt = 0$$

(because $F(h) = \int_2^{2+h}\sqrt{t^4+1}\,dt$ is a continuous function by Fundamental Theorem of Calculus Part I), and:

$$\lim_{h\to 0^+}\ln(1+h) = \ln(1+0) = 0$$

(also because $f(h) = \ln(1+h)$ is a continuous function).

Hence, the limit (13.1) *corresponds to the indeterminate form $\frac{0}{0}$.*

Taking the limit of the ratio of the derivatives of the numerator and denominator, we have:

$$\begin{aligned}\lim_{h\to 0^+}\frac{\frac{d}{dh}\int_2^{2+h}\sqrt{t^4+1}\,dt}{\frac{d}{dh}\ln(1+h)} &= \lim_{h\to 0^+}\frac{\left(\sqrt{(2+h)^4+1}\right)(2+h)'}{\frac{1}{1+h}} \\ &= \lim_{h\to 0^+}(1+h)\left(\sqrt{(2+h)^4+1}\right) \\ &= \sqrt{17}.\end{aligned}$$

It now follows from l'Hôpital's rule that:

$$\lim_{h\to 0^+} \frac{1}{\ln(1+h)} \int_2^{2+h} \sqrt{t^4+1}\,dt = \sqrt{17}.$$

There is a general formula regarding derviatives of the form:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)\,dt,$$

the discussion of which is beyond the scope of this course. However, in certain special cases, the derivative may be found using Corollary 13.9 without much further effort:

Example 13.12. *Find:*

$$\frac{d}{dx}\int_x^{3x^2} \frac{\sin(x^2 t)}{t}\,dt, \quad x > 0. \tag{13.2}$$

Again, we first view x as a constant.
Let:

$$u = x^2 t.$$

So:

$$t = \frac{u}{x^2}, \quad dt = \frac{1}{x^2}du.$$

Under this change of variable, the integral:

$$\int_{t=x}^{t=3x^2} \frac{\sin(x^2 t)}{t}\,dt$$

is equal to:

$$\int_{u=x^3}^{u=3x^4} \frac{\sin(u)}{(u/x^2)}\frac{1}{x^2}\,du = \int_{u=x^3}^{u=3x^4} \frac{\sin(u)}{u}\,du$$

It now follows from Corollary 13.9 that:

$$\begin{aligned}\frac{d}{dx}\int_{t=x}^{t=3x^2} \frac{\sin(x^2 t)}{t}\,dt &= \frac{d}{dx}\left[\int_{u=x^3}^{u=3x^4} \frac{\sin(u)}{u}\,du\right]. \\ &= \frac{\sin(3x^4)}{3x^4}\cdot 12x^3 - \frac{\sin(x^3)}{x^3}\cdot 3x^2 \\ &= \frac{4\sin(3x^4)}{x} - \frac{3\sin(x^3)}{x}.\end{aligned}$$

Theorem 13.13 (Fundamental Theorem of Calculus Part II)**.** *Let f be a continuous function on $[a, b]$. Let F be a continuous function on $[a, b]$ which is an antiderivative of f over (a, b). Then:*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof. By the Fundamental Theorem of Calculus Part I, we know that $G(x) = \int_a^x f(t)\,dt$ is also an antiderivative of f. By Lagrange's Mean Value Theorem and the continuity of F and G on $[a, b]$, for all $x \in [a, b]$ we have:

$$G(x) = F(x) + C$$

for some constant C.

Since $G(a) = \displaystyle\int_a^a f(t)\,dt = 0$, we have $C = -F(a)$.

Hence:

$$\int_a^b f(t)\,dt = G(b) = F(b) + C = F(b) - F(a).$$

□