Math 1010 Week 11

Indefinite Integrals, Integration of Trig. Functions, Trigonometric Substitution

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11.1 Integration of Trigonometric Functions

We have seen that:

$$\int \sin^2 x \, dx = \frac{x}{2} - \frac{1}{4}\sin(2x) + C$$

$$\int \cos^2 x \, dx = \frac{x}{2} + \frac{1}{4}\sin(2x) + C$$

Example 11.1. *Using:*

$$\int \sec^2 x \, dx = \tan x + C,$$
$$\int \csc^2 x \, dx = -\cot x + C,$$

and the identity $1 + \tan^2 x = \sec^2 x$ *(which follows from the Pythagorean Theorem), we may evaluate:*

- $\displaystyle\int \tan^2 x \, dx$

  $$\begin{aligned}\int \tan^2 x \, dx &= \int (\sec^2 x - 1) \, dx \\ &= \tan x - x + C,\end{aligned}$$

  where C *represents an arbitrary constant.*

- $\int \cot^2 x \, dx$

$$\begin{aligned} \int \cot^2 x \, dx &= \int (\csc^2 x - 1) \, dx \\ &= -\cot x - x + C, \end{aligned}$$

where C represents an arbitrary constant.

To evaluate an integral of the form:

$$\int \sin^m x \cos^n x \, dx, \quad n, m \in \mathbb{N},$$

it is useful to make the following substitution:

$$u = \begin{cases} \cos x, & \text{if } m \text{ is odd,} \\ \sin x, & \text{if } n \text{ is odd,} \end{cases}$$

and then apply the Pythagorean Theorem $\cos^2 x + \sin^2 x = 1$ to rewrite the original integral as:

$$\int P(u) \, du,$$

where $P(u)$ is some polynomial in u.

Example 11.2. *Evaluate:*

$$\int \cos^5 x \sin^3 x \, dx$$

$$\int \cos^5 x \sin^3 x \, dx = \int \cos^5 x \sin^2 x (\sin x \, dx)$$

Let $u = \cos x$. Then, $du = \sin x \, dx$. So,

$$\begin{aligned} \int \cos^5 x \sin^3 x \, dx &= \int \cos^5 x \sin^2 x (\sin x \, dx) \\ &= \int u^5 (1 - u^2) du \\ &= \int \left(u^5 - u^7\right) du \\ &= \frac{1}{6} u^6 - \frac{1}{8} u^8 + C \\ &= \frac{1}{6} \cos^6 x - \frac{1}{8} \cos^8 x + C, \end{aligned}$$

where C represents an arbitrary constant.

Similarly, to evaluate integrals of the form:

$$\int \tan^m x \sec^n x \, dx, \quad m, n \in \mathbb{N},$$

it is useful to make the following substitution:

$$u = \begin{cases} \sec x, & \text{if } m \text{ is odd,} \\ \tan x, & \text{if } n \text{ is even,} \end{cases}$$

and then apply the identity $1 + \tan^2 x = \sec^2 x$ to rewrite the original integral as:

$$\int P(u) \, du,$$

where $P(u)$ is some polynomial in u.

Example 11.3. *Evaluate:* $\int \tan^3 x \sec x \, dx$.

$$\begin{aligned} \int \tan^3 x \sec x \, dx &= \int \tan^2 x \sec x \tan x \, dx \\ &= \int (\sec^2 - 1) \, dx. \end{aligned}$$

Let $u = \sec x$. *Then,* $du = \sec x \tan x \, dx$, *and:*

$$\begin{aligned} \int \tan^3 x \sec x \, dx &= \int \tan^2 x \sec x \tan x \, dx \\ &= \int (\sec^2 - 1) \sec x \tan x \, dx \\ &= \int (u^2 - 1) \, du \\ &= \frac{1}{3} u^3 - u + C \\ &= \frac{1}{3} \sec^3 x - \sec x + C, \end{aligned}$$

where C *represents an arbitrary constant.*

Claim 11.4.

$$\int \sec x \, dx = \ln |\sec x + \tan x| + C,$$

where C *represents an arbitrary constant.*

Proof.

$$\begin{aligned}\int \sec x\,dx &= \int \frac{1}{\cos x}\,dx\\ &= \int \frac{\cos x}{\cos^2 x}\,dx\\ &= \int \frac{\cos x}{1-\sin^2 x}\,dx\end{aligned}$$

Let $u = \sin x$. Then $du = \cos x\,dx$, and consequently:

$$\begin{aligned}\int \sec x\,dx &= \int \frac{1}{1-u^2}\,du\\ &= \int \frac{1}{(1-u)(1+u)}\,du\\ &= \frac{1}{2}\int \left(\frac{1}{1-u} + \frac{1}{1+u}\right)du\\ &= \frac{1}{2}\left(-\ln|1-u| + \ln|1+u|\right) + C\\ &= \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| + C\\ &= \frac{1}{2}\ln\left|\frac{(1+u)^2}{1-u^2}\right| + C\\ &= \ln\left|\frac{1+u}{\sqrt{1-u^2}}\right| + C\\ &= \ln\left|\frac{1+\sin x}{\cos x}\right| + C\\ &= \ln|\sec x + \tan x| + C,\end{aligned}$$

where C represents an arbitrary constant. □

Example 11.5. *Evaluate:* $\int \sec^3 x\,dx$. *(Hint: Consider using integration by parts.)*

$$\int \sec^3 x\,dx = \int \sec x \sec^2 x\,dx.$$

Let $U = \sec x$*,* $dV = \sec^2 x\,dx$*. Taking* $V = \tan x$*, it follows from the Integration* *by Parts formula that:*

$$\begin{aligned}
\int \sec^3 x\, dx &= \int U dV \\
&= UV - \int V\, du \\
&= \sec x \tan x - \int \tan x \sec x \tan x\, dx \\
&= \sec x \tan x - \int \sec x \tan^2 x\, dx \\
&= \sec x \tan x - \int \sec x (\sec^2 x - 1)\, dx \\
&= \sec x \tan x - \int \left(\sec^3 x - \sec x\right)\, dx \\
&= \sec x \tan x + \ln|\sec x + \tan x| - \int \sec^3 x\, dx
\end{aligned}$$

This implies that:

$$2 \int \sec^3 x\, dx = \sec x \tan x + \ln|\sec x + \tan x| + C$$

where C represents an arbitrary constant. Hence:

$$\int \sec^3 x\, dx = \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C.$$

The following identities follow directly from the angle sum formulas of the sine and cosine functions:

$$\begin{aligned}
\cos x \cos y &= \frac{1}{2}\left(\cos(x+y) + \cos(x-y)\right) \\
\cos x \sin y &= \frac{1}{2}\left(\sin(x+y) - \sin(x-y)\right) \\
\sin x \sin y &= \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)
\end{aligned}$$

They are useful for the evaluation of integrals such as:

Example 11.6.

$$\int \cos(3x) \sin(5x)\, dx$$

$$
\begin{aligned}
\int \cos(3x)\sin(5x)\,dx &= \int \frac{1}{2}\left(\sin(3x+5x) - \sin(3x-5x)\right)\,dx \\
&= \frac{1}{2}\int \left(\sin(8x) + \sin(2x)\right)\,dx \\
&= \frac{1}{2}\left(-\frac{1}{8}\cos(8x) - \frac{1}{2}\cos(2x)\right) + C,
\end{aligned}
$$

where C represents an arbitrary constant.

11.2 WeBWorK

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11.3 Trigonometric Substitution

When an integrand involves $\sqrt{x^2 \pm a^2}$ or $\sqrt{a^2 - x^2}$. It is sometimes useful to make the following substitution:

- $\sqrt{x^2 + a^2}$: Let $x = a\tan\theta$.
- $\sqrt{x^2 - a^2}$: Let $x = a\sec\theta$.
- $\sqrt{a^2 - x^2}$: Let $x = a\sin\theta$.

Example 11.7. *Evaluate:* $\displaystyle\int \frac{x^3}{\sqrt{1-x^2}}\,dx$

First, we note that the domain of the integrand is $(-1,1)$.

Let $\theta = \arcsin x$. *Then* $x = \sin\theta$, $dx = \cos\theta\,d\theta$, *and:*

$$\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = |\cos\theta| = \cos\theta,$$

since $\theta = \arcsin x \in [-\pi/2, \pi/2]$ *for all* $x \in (-1,1)$.

So,

$$\begin{aligned}
\int \frac{x^3}{\sqrt{1-x^2}}\,dx &= \int \frac{\sin^3\theta}{\cos\theta}\cos\theta\,d\theta \\
&= \int \sin^3\theta\,d\theta \\
&= \int (1-\cos^2\theta)\sin\theta\,d\theta \\
&= -\int (1-\cos^2\theta)\,d(\cos\theta) \\
&= -\cos\theta + \frac{1}{3}\cos^3\theta + C \\
&= -\sqrt{1-x^2} + \frac{1}{3}(1-x^2)^{3/2} + C.
\end{aligned}$$

Example 11.8. *Evaluate:* $\displaystyle\int \frac{1}{(9+x^2)^2}\,dx$

Let $\theta = \arctan(x/3)$. *Then* $x = 3\tan\theta$, $dx = 3\sec^2\theta\,d\theta$, *and:*

$$9 + x^2 = 9 + 9\tan^2\theta = 9\sec^2\theta.$$

So,

$$\begin{aligned}
\int \frac{1}{(9+x^2)^2}\,dx &= \int \frac{1}{81\sec^4\theta} 3\sec^2\theta\,d\theta \\
&= \int \frac{1}{27\sec^2\theta}\,d\theta \\
&= \frac{1}{27}\int \cos^2\theta\,d\theta \\
&= \frac{1}{27}\left(\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right) + C \\
&= \frac{1}{27}\left(\frac{\theta}{2} + \frac{2\sin\theta\cos\theta}{4}\right) + C \\
&= \frac{1}{27}\left(\frac{\theta}{2} + \frac{2\tan\theta\cos^2\theta}{4}\right) + C
\end{aligned}$$

$$= \frac{\arctan(x/3)}{54} + \frac{\tan(\arctan(x/3))\cos^2(\arctan(x/3))}{54} + C$$

Now,

$$\begin{aligned}
\cos^2(\arctan(x/3)) &= \frac{1}{\sec^2(\arctan(x/3))} \\
&= \frac{1}{1+\tan^2(\arctan(x/3))} \\
&= \frac{1}{1+(x/3)^2} = \frac{9}{9+x^2}
\end{aligned}$$

Hence,

$$\begin{aligned}
\int \frac{1}{(9+x^2)^2}\,dx &= \frac{\arctan(x/3)}{54} + \frac{9x}{162(9+x^2)} + C \\
&= \frac{\arctan(x/3)}{54} + \frac{x}{18(9+x^2)} + C
\end{aligned}$$

Example 11.9. *Evaluate:* $\int \frac{\sqrt{x^2-25}}{x}\,dx$

Example 11.10. *Evaluate:* $\int \frac{x}{8-2x-x^2}\,dx.$

Example 11.11. *Evaluate:*

$$\int \frac{dx}{x\sqrt{x^2-1}}$$

First, we note that the domain of the integrand is $(-\infty,-1)\cup(1,\infty)$.

Let $\theta = \arccos(1/x)$.

Then, $x = \sec\theta$, $dx = \sec\theta\tan\theta\,d\theta$, *and:*

$$\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = \left|\tan\theta\right|.$$

Since:

$$\theta = \arccos(1/x) \in \begin{cases} [0,\pi/2) & \textit{if } x>1, \\ (\pi/2,\pi] & \textit{if } x<-1, \end{cases}$$

we have:

$$\sqrt{x^2-1} = |\tan\theta| = \begin{cases} \tan\theta & \textit{if } x>1, \\ -\tan\theta & \textit{if } x<-1. \end{cases}$$

More succinctly, we have:

$$\sqrt{x^2-1} = \operatorname{sign}(x)\tan\theta.$$

Hence,

$$\begin{aligned}\int \frac{dx}{x\sqrt{x^2-1}} &= \int \operatorname{sign}(x)\frac{\sec\theta\tan\theta}{\sec\theta\tan\theta}d\theta \\ &= \int \operatorname{sign}(x)\, d\theta \\ &= \operatorname{sign}(x)\theta + C \\ &= \operatorname{sign}(x)\arccos(1/x) + C\end{aligned}$$

11.4 WeBWorK

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