Math 1010 Week 3

Functions, Limits, Sandwich Theorem

3.1 Limits of Functions on the Real Line

Let $f : A \longrightarrow \mathbb{R}$ be a function, where $A \subseteq \mathbb{R}$. Let a be a point on the real line such that f is defined on a neighborhood of a (though not necessarily at a itself).

Definition 3.1. *We say that the* **limit** *of f at a is L if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever x satisfies $0 < |x - a| < \delta$.*

If f has a limit L at a, we write:

$$\lim_{x \to a} f(x) = L.$$

Note that the limit may exist even if a does not lie in the domain of f.

Remark. *Intuitively, $\lim_{x \to a} f(x) = L$ means that the value $f(x)$ approaches L as x approaches a from either side, or that $f(x)$ is very near L whenever x is very near a. Obviously, the term "near" is somewhat vague, and it is precisely because of this vagueness that mathematicians feel the need to define limits rigorously using the "δ-ε" language.*

Example 3.2. *Consider $f(x) = \dfrac{x^2 - 4}{x + 2}$. Note that the function f is not defined at -2.*

Observe that for x near -2, for example, $x = -2.001$, or $x = -1.9999$, we have:

$$f(-2.001) = -4.001,$$

$$f(-1.9999) = -3.9999,$$

which are close to -4.

Moreover, as x "approaches" -2 ($x = -2.001, -2.0001, -2.00001, \ldots$), we have $f(x) = -4.001, -4.0001, -4.00001$. So, it appears $f(x)$ approaches -4 as x approaches -2. This suggests that the limit of $f(x)$ at $x = -2$ is:

$$\lim_{x \to -2} f(x) = -4.$$

This turns out to be true, and is not surprising, since we can rewrite $f(x)$ as follows:

$$f(x) = \begin{cases} \dfrac{(x+2)(x-2)}{x+2}, & \text{if } x \neq -2; \\ \text{undefined}, & \text{if } x = -2. \end{cases}$$
$$= \begin{cases} x-2, & \text{if } x \neq -2; \\ \text{undefined}, & \text{if } x = -2. \end{cases}$$

Hence, all along we have really been asking what $x - 2$ tends to as x tends to -2.

Definition 3.3. *Let $f : A \longrightarrow \mathbb{R}$ be a function, where $A \subseteq \mathbb{R}$ is unbounded towards $+\infty$ and/or $-\infty$. We say that the* **limit** *of f at ∞ (resp. $-\infty$) is L if for all $\varepsilon > 0$, there exists a $c \in \mathbb{R}$ such that $|f(x) - L| < \varepsilon$ whenever $x > c$ (resp. $x < c$).*

If f has a limit L at ∞ (resp $-\infty$), we write:

$$\lim_{x \to \infty} f(x) = L \quad \left(\text{resp.} \lim_{x \to -\infty} f(x) = L\right)$$

3.1.1 Some Useful Identities

In the following idenities, the symbol a can be either a real number or $\pm\infty$.

1. For any constant $c \in \mathbb{R}$, we have $\lim\limits_{x \to a} c = c$.
2. $\lim\limits_{x \to a} x = a$.
3. If $\lim_{x \to a} f(x) = L$, and $\lim_{x \to a} g(x) = M$, then:
   - $\lim_{x \to a}(f \pm g)(x) = L \pm M$.
   - $\lim_{x \to a} fg(x) = LM$.
   - $$\lim_{x \to a} \frac{f}{g}(x) = \frac{L}{M}$$
     provided that $M \neq 0$.

4. If $\lim_{x\to a} f(x) = L$, then:

$$\lim_{x\to a}(f(x))^n = L^n \quad \text{for all} n \in \mathbb{N} = \{1, 2, 3, \ldots\},$$

and

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{L} \quad \text{for all odd positive integers} n.$$

In particular, for all positive integer n, we have:

$$\lim_{x\to a} x^n = a^n.$$

5. If $\lim_{x\to a} f(x) = L > 0$, then $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{L}$ for all $n \in \mathbb{N}$.

Example 3.4. *Compute the following limits, if they exist:*

- $\lim_{x\to -1} \frac{x^2 - 1}{x^2 - 5x - 6}$

- $\lim_{x\to 4} \frac{2 - \sqrt{x}}{16 - x^2}$

3.2 WeBWorK

1. **WeBWorK**
2. **WeBWorK**
3. **WeBWorK**
4. **WeBWorK**
5. **WeBWorK**

3.3 One-Sided Limits

- We write $\lim_{x\to a^+} f(x) = L$ if $f(x)$ approaches L as x approaches a *from the right* . We call this L the **right limit** of f at a.
- Similarly, we write $\lim_{x\to a^-} f(x) = L$ if $f(x)$ approaches L as x approaches a *from the left* . We call this L the **left limit** of f at a.

The limit $\lim_{x \to a} f(x)$ is sometimes called the **double-sided limit** of f at a. It exists if and only if both one-sided limits exist and are equal to each other. In which case, we have:

$$\lim_{x \to a} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x).$$

Exercise 3.5. *Define*

$$f(x) = \begin{cases} x - 1 & \textit{if } 1 \le x \le 2, \\ 2x + 3 & \textit{if } 2 < x \le 4, \\ x^2 & \textit{otherwise.} \end{cases}$$

Compute $\lim_{x \to 2^+} f(x)$ *and* $\lim_{x \to 2^-} f(x)$. *Then, find* $\lim_{x \to 2} f(x)$, *if it exists.*

Answers.

1.

$$\lim_{x \to 2^+} f(x) = 7$$
$$\lim_{x \to 2^-} f(x) = 1$$

2. *Since* $\lim_{x \to 2^+} f(x) \neq \lim_{x \to 2^-} f(x)$, *the double-sided limit* $\lim_{x \to 2} f(x)$ *does not exist.*

3.4 WeBWorK

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3.5 Sandwich Theorem for Functions on the Real Line

Theorem 3.6. *Let* $a \in \mathbb{R}$, A *an open neighborhood of* a *which does not necessarily contain* a *itself. Let* $f, g, h : A \longrightarrow \mathbb{R}$ *be functions such that:*

$$g(x) \le f(x) \le h(x) \quad \textit{for all} x \in A,$$

and

$$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L.$$

Then, $\lim_{x\to a} f(x) = L$.

Similary,

Theorem 3.7. *If f, g, h are functions on $\mathbb{R}$ such that:*

$$g(x) \leq f(x) \leq h(x)$$

for all x sufficiently large, and

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} h(x) = L,$$

then $\lim_{x\to\infty} f(x) = L$.

Exercise 3.8. *Find the following limits, if they exist:*

- $\lim_{x\to\infty} \frac{\sin x}{x}$

- $\lim_{x\to\infty} \frac{x+\sin x}{x-\sin x}$

Theorem 3.9.

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Corollary 3.10.

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}\ .$$

Proof.

$$\begin{aligned}
\lim_{x\to 0} \frac{1-\cos x}{x^2} &= \lim_{x\to 0} \frac{1-\cos x}{x^2} \cdot \left(\frac{1+\cos x}{1+\cos x}\right) \\
&= \lim_{x\to 0} \frac{1-\cos^2 x}{x^2\,(1+\cos x)} \\
&= \lim_{x\to 0} \frac{\sin^2 x}{x^2\,(1+\cos x)} \\
&= \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2 \frac{1}{1+\cos x} \\
&= \left(\lim_{x\to 0} \frac{\sin x}{x}\right)^2 \cdot \left(\lim_{x\to 0} \frac{1}{1+\cos x}\right) \\
&= 1^2 \cdot \frac{1}{1+1} = \frac{1}{2}
\end{aligned}$$

$\square$

Corollary 3.11.

$$\lim_{x\to 0}\frac{1-\cos x}{x}=0\ .$$

Exercise 3.12. *Find the following limits, if they exist:*

- $\lim_{x\to 0}\frac{\sin(5x)}{\tan(3x)}$

- $\lim_{x\to 0}\frac{x^3\cos\left(\frac{1}{x}\right)}{\tan x}$

3.6 WeBWorK

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4. **WeBWorK**
5. **WeBWorK**
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