# MATH 1030 Chapter 5

The lecture is based on Beezer, A first course in Linear algebra. Ver 3.5 Downloadable at http://linear.ups.edu/download.html .

The print version can be downloaded at http://linear.ups.edu/download/fcla-3.50-print.pdf .

**Reference.**

Beezer, Ver 3.5 Subsection RREF (print version p21 - p33) You can skip the proof of Thm REMEF on p.22 and Thm RREFU on p.24-27

**Exercise.**

Exercises with solutions can be downloaded at http://linear.ups.edu/download/fcla-3.50-solution-manual.pdf Section SSLE (p.1-6) C30-34, C50, M30, T20. Sect RREF (p.6-13) C10-19, C31-33, M40 Part 1, T10, T11, T12.

## 5.1 Reduced Row Echelon Form

**Terminology** :

- **Zero row**: A row consisting only of $0$'s.
- **Leftmost nonzero entry of a row**: The first nonzero entry of a row.
- **Index of the leftmost nonzero entry of a row**: The column index of the first nonzero entry in the row.

**Notation** : Denote by $d_i$ the index of leftmost nonzero entry of row $i$.

**Example 5.1.** The underlined entries are the leftmost nonzero entry for each row.

$$\begin{bmatrix} 0 & \underline{1} & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & \underline{1} \\ 0 & 0 & 0 & \underline{1} & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The index of the leftmost nonzero entry of row 1 is $d_1 = 2$. The index of the leftmost nonzero entry of row 2 is $d_2 = 5$. The index of the leftmost nonzero entry of row 3 is $d_3 = 4$. row 4 is a zero row.

**Example 5.2.** The underlined entries are the leftmost nonzero entry for each row.

$$\begin{bmatrix} \underline{2} & 0 & 1 & 2 & 3 & 4 \\ 0 & \underline{1} & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 & \underline{1} & 0 \\ 0 & \underline{-1} & 0 & 0 & 0 & 1 \end{bmatrix}$$

The index of the leftmost nonzero entry of row 1 is $d_1 = 1$. The index of the leftmost nonzero entry of row 2 is $d_2 = 2$. The index of the leftmost nonzero entry of row 3 is $d_3 = 5$. The index of the leftmost nonzero entry of row 3 is $d_4 = 2$.

A matrix is said to be in **reduced row echelon form** if it looks like this ( $*$ means an arbitary number)

$$\begin{bmatrix} 1 & * & \cdots & 0 & * & \cdots & 0 & * & \cdots \\ 0 & 0 & \cdots & 1 & * & \cdots & 0 & * & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

More precisely:

1. It looks like an inverted staircase.
2. Each new step down (i.e. up) gives a leading "1". Above it are $0$'s.
3. The column that is at the edge a new step is call a **pivot column**.

**Definition 5.3** (Reduced Row-Echelon Form)**.** A matrix is in **reduced row-echelon form** if it meets all of the following conditions:

1. If there is a row where every entry is zero, then this row lies below any other row that contains a nonzero entry.
2. The leftmost nonzero entry of a nonzero row is equal to 1.
3. The leftmost nonzero entry of a nonzero row is the only nonzero entry in its column.
4. If $i < j$ and both row $i$ and row $j$ are not zero rows, then $d_i < d_j$, i.e. $d_1, d_2, \ldots$ are in ascending order.

   In particular, all matrix entries below a leftmost nonzero entry must be equal to zero.

**Terminology.** A row of only zero entries is called a **zero row**.

In the case of a matrix in reduced row-echelon form, the leftmost nonzero entry of a nonzero row is a **leading 1**.

A column containing a leading 1 will be called a **pivot column**.

The number of nonzero rows will be denoted by $r$, which is also equal to the number of leading 1's and the number of pivot columns.

The set of column indices for the pivot columns will be denoted by:

$$D = \{d_1,\, d_2,\, d_3,\, \ldots,\, d_r\},$$

where:

$$d_1 < d_2 < d_3 < \cdots < d_r.$$

The columns that are not pivot columns will be denoted as:

$$F = \{f_1,\, f_2,\, f_3,\, \ldots,\, f_{n-r}\},$$

where:

$$f_1 < f_2 < f_3 < \cdots < f_{n-r}.$$

**Example 5.4.** The matrix below are in reduced row echelon from

1.
$$\begin{bmatrix} 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   Definition 5.3 (Reduced Row-Echelon Form)

   Column 1, 3, 6 are pivot columns, $r = 3$, $D = \{1, 3, 6\}$, $d_1 = 1, d_2 = 3, d_3 = 6$, $F = \{2, 4, 5\}$, $f_1 = 2, f_2 = 4, f_3 = 5$.

2.
$$\begin{bmatrix} 1 & 0 & 5 & 3 & 0 & 0 & 5 \\ 0 & 1 & 3 & 6 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

   Definition 5.3 (Reduced Row-Echelon Form)

   Column 1, 2, 5, 6 are pivot columns, $r = 4$, $D = \{1, 2, 5, 6\}$, $d_1 = 1, d_2 = 2, d_3 = 5, d_4 = 6$, $F = \{3, 4, 7\}$, $f_1 = 3, f_2 = 4, f_3 = 7$.

3.
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Definition 5.3 (Reduced Row-Echelon Form)

Column 1, 2, 3 are pivot columns, $r = 3$, $D = \{1, 2, 3\}$, $d_1 = 1, d_2 = 2, d_3 = 3$, $F = \emptyset$ (an empty set).

4.
$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 0 & 9 & 6 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition 5.3 (Reduced Row-Echelon Form)

Column 2, 5, 7 are pivot columns. Note that column 3 is **not** a pivot column. $r = 3$, $D = \{2, 5, 7\}$, $d_1 = 2, d_2 = 5, d_3 = 7$, $F = \{1, 3, 4, 6, 8, 9\}$, $f_1 = 1, f_2 = 3, f_3 = 4, f_4 = 6, f_5 = 8, f_6 = 9$.

5. The matrix $C$ is in reduced row-echelon form.

$$C = \begin{bmatrix} 1 & -3 & 0 & 6 & 0 & 0 & -5 & 9 \\ 0 & 0 & 0 & 0 & 1 & 0 & 3 & -7 \\ 0 & 0 & 0 & 0 & 0 & 1 & 7 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition 5.3 (Reduced Row-Echelon Form)

This matrix has two zero rows and three pivot columns. So $r = 3$. Columns 1, 5, and 6 are the pivot columns, so $D = \{1,\ 5,\ 6\}$, $d_1 = 1, d_2 = 5, d_3 = 6$, $F = \{2,\ 3,\ 4,\ 7,\ 8\}$, $f_1 = 2, f_2 = 3, f_3 = 4, f_4 = 7, f_5 = 8$.

**Example 5.5.** The following matrices are **not RREF**, explain why.

1.
$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It fails condition 1: row 3 is a zero row but row 4, which is under row 3, is not a zero row.

2. The underline entries are the leftmost nonzero entry for each row.

$$\begin{bmatrix} \underline{1} & 0 & 2 & 0 \\ 0 & \underline{1} & 3 & 0 \\ 0 & 0 & 0 & \underline{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   It fails condition 2: the leftmost nonzero entry of row 3 is not 1.

3. The underline entries are the leftmost nonzero entry for each row.

$$\begin{bmatrix} 0 & \underline{1} & 0 & 0 & 1 & 2 \\ 0 & 0 & \underline{1} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \underline{1} & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   It fails condition 3: For row 3, the column consists the left most nonzero entry (i.e. column 5) has more than $1$ nonzero entries.

4. The underline entries are the leftmost nonzero entry for each row.

$$\begin{bmatrix} \underline{1} & 0 & 0 & 1 & 2 \\ 0 & 0 & \underline{1} & 0 & 1 \\ 0 & \underline{1} & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   It fails condition 4: The index of the leftmost nonzero entry of row 1 is $d_1 = 1$. The index of the leftmost nonzero entry of row 2 is $d_2 = 3$. The index of the leftmost nonzero entry of row 3 is $d_2 = 2$. $2 < 3$ but $d_2 > d_3$.

**Theorem 5.6** (Row-Equivalent Matrix in Echelon Form)**.** *Suppose $A$ is a matrix. Then there is a matrix $B$ such that:*

*1. $A$ and $B$ are row-equivalent.*

*2. $B$ is in reduced row-echelon form.*

*Proof of Row-Equivalent Matrix in Echelon Form.* Suppose that $A$ has $m$ rows and $n$ columns. We will describe a process for converting $A$ into $B$ via row operations. This procedure is known as **Gaussian elimination** or sometimes called **Gauss-Jordan elimination**. Tracing through this procedure will be easier if you recognize that $i$ refers to a row that is being converted, $j$ refers to a column that is being converted, and $r$ keeps track of the number of nonzero rows.

1. Set $j = 0$ and $r = 0$.
2. Increase $j$ by 1. If $j$ now equals $n + 1$, then stop.
3. Examine the entries of $A$ in column $j$ located in rows $r + 1$ through $m$. If all of these entries are zero, then go to Step 2.
4. Choose a row from rows $r + 1$ through $m$ with a nonzero entry in column $j$. Let $i$ denote the index for this row.
5. Increase $r$ by 1.
6. Use the first row operation to swap rows $i$ and $r$.
7. Use the second row operation to convert the entry in row $r$ and column $j$ to a 1.
8. Use the third row operation with row $r$ to convert every other entry of column $j$ to zero.
9. Go to Step 2.

The result of this procedure is that the matrix $A$ is converted to a matrix in reduced row-echelon form, which we will refer to as $B$. he matrix is only converted through row operations (Steps 6, 7, 8), so $A$ and $B$ are row-equivalent. We need to now prove this claim by showing that the converted matrix has the requisite properties of Definition 5.3 (Reduced Row-Echelon Form). We will skip the proof for now. See Beezer, Ver 3.5 (print version p23). □

We will now run through some examples of using these definitions and theorems to solve some systems of equations. From now on, when we have a matrix in reduced row-echelon form, we will mark the leading 1's with a small box.

**Example 5.7.** Using the Gaussian elimination, find the RREF of

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 1 & 1 & 2 & 4 & 8 \\ 2 & 2 & 5 & 9 & 19 \end{bmatrix}$$

Set $r = 0$. Consider column 1 (set $j = 1$), find a nonzero entry (underline below) in the column.

$$\begin{bmatrix} \underline{0} & 0 & 1 & 1 & 4 \\ \underline{0} & 0 & 1 & 1 & 3 \\ \underline{1} & 1 & 2 & 4 & 8 \\ \underline{2} & 2 & 5 & 9 & 19 \end{bmatrix}$$

Move the nonzero entry to row 1 by swapping rows $R_1 \leftrightarrow R_i$.

If the entry at row 1, column 1 is nonzero, you don't have to swap rows. But you can consider swap it with entry $= 1$ or $-1$.

In this example, for column 1, 3rd entry and 4th entry are nonzero, so we can use $R_1 \leftrightarrow R_3$ or $R_1 \leftrightarrow R_4$.

There is nothing wrong about $R_1 \leftrightarrow R_4$ but **it is better to swap with the row with leading entry equal to $1$ or $-1$.**

So we use $R_1 \leftrightarrow R_3$.

$$\xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} \boxed{1} & 1 & 2 & 4 & 8 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 & 4 \\ 2 & 2 & 5 & 9 & 19 \end{bmatrix}$$

Also, at this point we increase $r$ to $r = 1$, since we now know we have at least one nonzero row.

If the boxed number is $1$, we are good to go.

If the boxed number is not equal to $1$, say it is $a$, use $\frac{1}{a}R_1$ to convert it to $1$.

Then use the boxed number to eliminate the nonzero entries above and below it by $\alpha R_1 + R_i$.

In our example, the boxed number is $1$, so we don't have to do anything. Use $-2R_1 + R_4$ and to remove the nonzero entries below it and above it. Since we are at the first row, so there is nothing above it).

$$\xrightarrow{-2R_1 + R_4} \begin{bmatrix} \boxed{1} & 1 & 2 & 4 & 8 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 1 & 3 \end{bmatrix}$$

Now we go back to Step 1 in the proof of Theorem 5.6 (Row-Equivalent Matrix in Echelon Form), with column index $j$ increased to 2 and $r = 1$.

In fact, we may as well ignore row 1 and column 1, and essentially apply the previous steps to the remaining $3 \times 4$ matrix:

$$\begin{bmatrix} * & * & * & * & * \\ * & \underline{0} & 1 & 1 & 3 \\ * & \underline{0} & 1 & 1 & 4 \\ * & \underline{0} & 1 & 1 & 3 \end{bmatrix}$$

The entries of column 2 are underlined.

None of them are nonzero, so we move to next column.

$$\begin{bmatrix} * & * & * & * & * \\ * & 0 & \underline{1} & 1 & 3 \\ * & 0 & \underline{1} & 1 & 4 \\ * & 0 & \underline{1} & 1 & 3 \end{bmatrix}$$

Consider column 3. That is, set $j = 3$. (Note that the number of nonzero rows is still $r = 1$.)

All the entries of column 3 are equal to 1. So, we don't need to do any swapping.

$$\begin{bmatrix} 1 & 1 & 2 & 4 & 8 \\ 0 & 0 & \boxed{1} & 1 & 3 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 1 & 3 \end{bmatrix}$$

Also, now we know there are at least 2 nonzero rows, so $r = 2$. Use the boxed number (the pivot) to eliminate the nonzero entries above and below it by $\alpha R_2 + R_i$.

$$\xrightarrow{\substack{-2R_2 + R_1, \\ -1R_2 + R_3, \\ -1R_2 + R_4}} \begin{bmatrix} 1 & 1 & 0 & 2 & 2 \\ 0 & 0 & \boxed{1} & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, we may ignore rows 2 and 3, and columns 1 through 3. With $r = 2$, and the column index increased to $j = 4$, we repeat the whole process.

$$\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & \underline{0} & 1 \\ * & * & * & \underline{0} & 0 \end{bmatrix}$$

All the underlined entries are zeros, so we move to the next row.

$$\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & 0 & \underline{1} \\ * & * & * & 0 & \underline{0} \end{bmatrix}$$

We can then use the boxed number to eliminate all the nonzero entries above it and below it and get the RREF.

$$\begin{bmatrix} 1 & 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-2R_3+R_1,\ -3R_3+R_2} \begin{bmatrix} 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**Example 5.8.** Using the Gaussian elimination, find the RREF of

$$A = \begin{bmatrix} 0 & 0 & 2 & 2 & 6 & 2 & 3 \\ 2 & 4 & 1 & 3 & 7 & 3 & -1 \\ 1 & 2 & 2 & 3 & 8 & 2 & 1 \\ 1 & 2 & -1 & 0 & -1 & 2 & -1 \end{bmatrix}$$

Set $r = 0$. We first consider column 1 (set $j = 1$).

Find a nonzero entry (underline below) in the column.

$$\begin{bmatrix} \underline{0} & 0 & 2 & 2 & 6 & 2 & 3 \\ \underline{2} & 4 & 1 & 3 & 7 & 3 & -1 \\ \underline{1} & 2 & 2 & 3 & 8 & 2 & 1 \\ \underline{1} & 2 & -1 & 0 & -1 & 2 & -1 \end{bmatrix}$$

Move the nonzero entry to row 1 by swapping rows $R_1 \leftrightarrow R_i$.

If the entry at row 1, column 1 is nonzero, you don't have to swap rows. But you can consider swap it with entry $= 1$ or $-1$.

In this example, for column 1, 2nd entry, 3rd entry and 4th entry are nonzeros, so we can use $R_1 \leftrightarrow R_2$, $R_1 \leftrightarrow R_3$ or $R_1 \leftrightarrow R_4$.

There is nothing wrong about $R_1 \leftrightarrow R_2$ but **it is better to swap with the row with entry equal to $1$ or $-1$.**

So we use $R_1 \leftrightarrow R_3$.

$$\xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} \boxed{1} & 2 & 2 & 3 & 8 & 2 & 1 \\ 2 & 4 & 1 & 3 & 7 & 3 & -1 \\ 0 & 0 & 2 & 2 & 6 & 2 & 3 \\ 1 & 2 & -1 & 0 & -1 & 2 & -1 \end{bmatrix}$$

If the boxed number is 1, we are good to go.

If the boxed number is not equal to 1, say it is $a$, use $\frac{1}{a}R_1$ to convert it to 1.

Then use the boxed number to eliminate the nonzero entries above and below it by $\alpha R_1 + R_i$.

In our example, the boxed number is 1, so we don't have to do anything. Use $-2R_1 + R_2$ and $-1R_1 + R_4$ to remove the nonzero entries below it and above it. Since we are at the first row, so there is nothing above it.

$$\xrightarrow{-2R_1+R_2,\ -1R_1+R_4} \begin{bmatrix} 1 & 2 & 2 & 3 & 8 & 2 & 1 \\ 0 & 0 & -3 & -3 & -9 & -1 & -3 \\ 0 & 0 & 2 & 2 & 6 & 2 & 3 \\ 0 & 0 & -3 & -3 & -9 & 0 & -2 \end{bmatrix}$$

Ignore the row 1 and col 1. Consider column 2:

$$\begin{bmatrix} * & * & * & * & * & * & * \\ * & \underline{0} & -3 & -3 & -9 & -1 & -3 \\ * & \underline{0} & 2 & 2 & 6 & 2 & 3 \\ * & \underline{0} & -3 & -3 & -9 & 0 & -2 \end{bmatrix}$$

None of the entries of column 2 are nonzero.

So, we consider the next column ($j = 3$, $r = 1$).

$$\begin{bmatrix} * & * & * & * & * & * & * \\ * & 0 & \underline{-3} & -3 & -9 & -1 & -3 \\ * & 0 & \underline{2} & 2 & 6 & 2 & 3 \\ * & 0 & \underline{-3} & -3 & -9 & 0 & -2 \end{bmatrix}$$

Find a nonzero entry in column 3. In this case, all the entries are nonzero, so we may increase the number of nonzero rows to $r = 2$.

There is no entry equal to 1 or $-1$. We don't need to so any swapping.

$$\begin{bmatrix} * & * & * & * & * & * & * \\ * & 0 & \boxed{-3} & -3 & -9 & -1 & -3 \\ * & 0 & 2 & 2 & 6 & 2 & 3 \\ * & 0 & -3 & -3 & -9 & 0 & -2 \end{bmatrix}$$

Turn the boxed number into 1 by $-\frac{1}{3}R_2$.

$$\xrightarrow{-\frac{1}{3}R_2} \begin{bmatrix} 1 & 2 & 2 & 3 & 8 & 2 & 1 \\ 0 & 0 & \boxed{1} & 1 & 3 & \frac{1}{3} & 1 \\ 0 & 0 & 2 & 2 & 6 & 2 & 3 \\ 0 & 0 & -3 & -3 & -9 & 0 & -2 \end{bmatrix}$$

We then use the boxed number to remove the nonzero entries about it and below it.

$$\xrightarrow{-2R_2+R_1,\ -2R_2+R_3,\ 3R_2+R_4} \begin{bmatrix} 1 & 2 & 0 & 1 & 2 & \frac{4}{3} & -1 \\ 0 & 0 & 1 & 1 & 3 & \frac{1}{3} & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{4}{3} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Now, ignore the first 2 rows and the first 3 columns.

$$\begin{bmatrix} * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & \underline{0} & 0 & \frac{4}{3} & 1 \\ * & * & * & \underline{0} & 0 & 1 & 1 \end{bmatrix}$$

Consider the column with index $j = 4$.

All the entries in column 4 (underlined) are equal to zero. So, we move to the next column, with index $j = 5$.

$$\begin{bmatrix} * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & 0 & \underline{0} & \frac{4}{3} & 1 \\ * & * & * & 0 & \underline{0} & 1 & 1 \end{bmatrix}$$

Again, all the entries in column 5 (underlined) are equal to zero. So, we move to the next column, with index $j = 6$.

$$\begin{bmatrix} * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & 0 & 0 & \underline{\frac{4}{3}} & 1 \\ * & * & * & 0 & 0 & \underline{1} & 1 \end{bmatrix}$$

We continue the process without detailed explanations:

$$\xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 2 & 0 & 1 & 2 & \frac{4}{3} & -1 \\ 0 & 0 & 1 & 1 & 3 & \frac{1}{3} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{4}{3} & 1 \end{bmatrix}$$

$$\xrightarrow[-\frac{4}{3}R_3 + R_4]{-\frac{4}{3}R_3 + R_1,\ -\frac{1}{3}R_3 + R_2,} \begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 0 & -\frac{7}{3} \\ 0 & 0 & 1 & 1 & 3 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} \end{bmatrix}$$

$$\xrightarrow{-3R_4} \begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 0 & -\frac{7}{3} \\ 0 & 0 & 1 & 1 & 3 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[-1R_4 + R_3]{\frac{7}{3}R_4 + R_1,\ -\frac{2}{3}R_4 + R_2,} \begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = B$$

The matrix $B$ is the reduced row echelon form of $A$. We write:

$$A \xrightarrow{\text{RREF}} B.$$

**Theorem 5.9** (Reduced Row-Echelon Form is Unique)**.** *Suppose that $A$ is an $m \times n$ matrix and that $B$ and $C$ are $m \times n$ matrices that are row-equivalent to $A$ and in reduced row-echelon form. Then $B = C$.*

*Proof of Reduced Row-Echelon Form is Unique.* See Beezer, Ver 3.5 (print version p24). We will prove it later. You can skip the proof for now. □

**Example 5.10.** Find the solutions to the following system of equations,

$$\begin{aligned} -7x_1 - 6x_2 - 12x_3 &= -33 \\ 5x_1 + 5x_2 + 7x_3 &= 24 \\ x_1 + 4x_3 &= 5 \end{aligned}$$

First, form the augmented matrix, is

$$\left[\begin{array}{ccc|c} -7 & -6 & -12 & -33 \\ 5 & 5 & 7 & 24 \\ 1 & 0 & 4 & 5 \end{array}\right]$$

and work to reduced row-echelon form, first with $j = 1$,

$$\xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & 4 & 5 \\ 5 & 5 & 7 & 24 \\ -7 & -6 & -12 & -33 \end{array}\right] \xrightarrow{-5R_1+R_2} \left[\begin{array}{ccc|c} 1 & 0 & 4 & 5 \\ 0 & 5 & -13 & -1 \\ -7 & -6 & -12 & -33 \end{array}\right]$$

$$\xrightarrow{7R_1+R_3} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 4 & 5 \\ 0 & 5 & -13 & -1 \\ 0 & -6 & 16 & 2 \end{array}\right]$$

Now, with $j = 2$,

$$\xrightarrow{\frac{1}{5}R_2} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 4 & 5 \\ 0 & 1 & \frac{-13}{5} & \frac{-1}{5} \\ 0 & -6 & 16 & 2 \end{array}\right] \xrightarrow{6R_2+R_3} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 4 & 5 \\ 0 & \boxed{1} & \frac{-13}{5} & \frac{-1}{5} \\ 0 & 0 & \frac{2}{5} & \frac{4}{5} \end{array}\right]$$

And finally, with $j = 3$,

$$\xrightarrow{\frac{5}{2}R_3} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 4 & 5 \\ 0 & \boxed{1} & \frac{-13}{5} & \frac{-1}{5} \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{\frac{13}{5}R_3+R_2} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 4 & 5 \\ 0 & \boxed{1} & 0 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{-4R_3+R_1} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & -3 \\ 0 & \boxed{1} & 0 & 5 \\ 0 & 0 & \boxed{1} & 2 \end{array}\right]$$

This is now the augmented matrix of a very simple system of equations, namely $x_1 = -3$, $x_2 = 5$, $x_3 = 2$, which has an obvious solution. Furthermore, we can see that this is the **only** solution to this system, so we have determined the entire solution set,

$$S = \left\{ \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix} \right\}$$

**Example 5.11.** Let us find the solutions to the following system of equations,

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ 2x_1 + x_2 + x_3 &= 8 \\ x_1 + x_2 &= 5 \end{aligned}$$

First, form the augmented matrix,

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 1 & 1 & 8 \\ 1 & 1 & 0 & 5 \end{array}\right]$$

$$\xrightarrow{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 3 & -3 & 6 \\ 1 & 1 & 0 & 5 \end{array}\right] \xrightarrow{-1R_1+R_3} \left[\begin{array}{ccc|c} \boxed{1} & -1 & 2 & 1 \\ 0 & 3 & -3 & 6 \\ 0 & 2 & -2 & 4 \end{array}\right]$$

Now, with $j = 2$,

$$\xrightarrow{\frac{1}{3}R_2} \left[\begin{array}{ccc|c} \boxed{1} & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & -2 & 4 \end{array}\right] \xrightarrow{1R_2+R_1} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & -2 & 4 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_3} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 3 \\ 0 & \boxed{1} & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system of equations represented by this augmented matrix needs to be considered a bit differently than the previous case. First, the last row of the matrix is the equation $0 = 0$, which is always true, so it imposes no restrictions on our possible solutions and therefore we can safely ignore it as we analyze the other two equations. These equations are:

$$x_1 + x_3 = 3$$
$$x_2 - x_3 = 2.$$

While this system is fairly easy to solve, it also appears to have a multitude of solutions. For example, choose $x_3 = 1$ and see that then $x_1 = 2$ and $x_2 = 3$ will together form a solution. Or choose $x_3 = 0$, and then discover that $x_1 = 3$ and $x_2 = 2$ lead to a solution. Try it yourself: pick any value of $x_3$ you please, and figure out what $x_1$ and $x_2$ should be to make the first and second equations (respectively) true. We'll wait while you do that. Because of this behavior, we say that $x_3$ is a **free** or **independent** variable. But why do we vary $x_3$ and not some other variable? For now, notice that the third column of the augmented matrix is not a pivot column. With this idea, we can rearrange the two equations, solving each for the variable whose index is the same as the column index of a pivot column.

$$x_1 = 3 - x_3$$
$$x_2 = 2 + x_3$$

To write the set of solution vectors in set notation, we have:

$$S = \left\{ \begin{bmatrix} 3 - x_3 \\ 2 + x_3 \\ x_3 \end{bmatrix} \middle| \, x_3 \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \middle| \, x_3 \in \mathbb{R} \right\}$$

We will learn more in the next lecture about systems with infinitely many solutions and how to express their solution sets.

**Example 5.12.** Let us find the solutions to the following system of equations,

$$\begin{aligned} 2x_1 + x_2 + 7x_3 - 7x_4 &= 2 \\ -3x_1 + 4x_2 - 5x_3 - 6x_4 &= 3 \\ x_1 + x_2 + 4x_3 - 5x_4 &= 2 \end{aligned}$$

First, form the augmented matrix,

$$\left[\begin{array}{cccc|c} 2 & 1 & 7 & -7 & 2 \\ -3 & 4 & -5 & -6 & 3 \\ 1 & 1 & 4 & -5 & 2 \end{array}\right]$$

and work to reduced row-echelon form, first with $j = 1$,

$$\xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 1 & 4 & -5 & 2 \\ -3 & 4 & -5 & -6 & 3 \\ 2 & 1 & 7 & -7 & 2 \end{array}\right] \xrightarrow{3R_1 + R_2} \left[\begin{array}{cccc|c} 1 & 1 & 4 & -5 & 2 \\ 0 & 7 & 7 & -21 & 9 \\ 2 & 1 & 7 & -7 & 2 \end{array}\right]$$

$$\xrightarrow{-2R_1+R_3} \left[\begin{array}{cccc|c} \boxed{1} & 1 & 4 & -5 & 2 \\ 0 & 7 & 7 & -21 & 9 \\ 0 & -1 & -1 & 3 & -2 \end{array}\right]$$

Now, with $j = 2$,

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|c} \boxed{1} & 1 & 4 & -5 & 2 \\ 0 & -1 & -1 & 3 & -2 \\ 0 & 7 & 7 & -21 & 9 \end{array}\right] \xrightarrow{-1R_2} \left[\begin{array}{cccc|c} \boxed{1} & 1 & 4 & -5 & 2 \\ 0 & 1 & 1 & -3 & 2 \\ 0 & 7 & 7 & -21 & 9 \end{array}\right]$$

$$\xrightarrow{-1R_2+R_1} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 3 & -2 & 0 \\ 0 & 1 & 1 & -3 & 2 \\ 0 & 7 & 7 & -21 & 9 \end{array}\right] \xrightarrow{-7R_2+R_3} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 3 & -2 & 0 \\ 0 & \boxed{1} & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 & -5 \end{array}\right]$$

And finally, with $j = 4$,

$$\xrightarrow{-\frac{1}{5}R_3} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 3 & -2 & 0 \\ 0 & \boxed{1} & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-2R_3+R_2} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 3 & -2 & 0 \\ 0 & \boxed{1} & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{array}\right]$$

The third equation will read $0 = 1$. This is patently false, all the time. No choice of values for our variables will ever make it true. We are done. Since we cannot even make the last equation true, we have no hope of making all of the equations simultaneously true. So this system has **no solutions**, and its solution set is the empty set, $\varnothing = \{\ \}$ Notice that we could have reached this conclusion sooner. After performing the row operation $-7R_2 + R_3$, we can see that the third equation reads $0 = -5$, a false statement. Since the system represented by this matrix has no solutions, none of the systems represented has any solutions. However, for this example, we have chosen to bring the matrix all the way to reduced row-echelon form as practice. The above three examples illustrate the full range of possibilities for a system of linear equations – no solutions, one solution, or infinitely many solutions. In the next lecture we will examine these three scenarios more closely.

## 5.2 Consistent Systems

A system of linear equations is **consistent** if it has at least one solution. Otherwise, the system is called **inconsistent**.

**Example 5.13.** 1. The system of linear equations

$$\begin{aligned} 2x_1 + 3x_2 &= 3 \\ x_1 - x_2 &= 4 \end{aligned}$$

is consistent because it has solution $(x_1, x_2) = (1, -3)$.

2. The system of linear equations

$$\begin{aligned} 2x_1 + 3x_2 &= 3 \\ 4x_1 + 6x_2 &= 6 \end{aligned}$$

is consistent because has infinite many solutions: $\{(t, \frac{3-2t}{3}) \mid t \text{ real number}\}$.

3. The system of linear equations

$$\begin{aligned} 2x_1 + 3x_2 &= 3 \\ 4x_1 + 6x_2 &= 10 \end{aligned}$$

is inconsistent because it has no solution.

**Notation**: Let $A$ be a reduced row echelon form.

1. The number of non-zero rows is called the **rank** of $A$ and is denoted by $r$.

2. The set of the column indexes for the pivot columns is denoted by

$$D = \{d_1, d_2, d_3, \ldots, d_r\},$$

where $d_1 < d_2 < d_3 < \cdots < d_r$.

3. The set of column indexes that are not pivot columns is denoted by

$$F = \{f_1, f_2, f_3, \ldots, f_{n-r}\},$$

where $f_1 < f_2 < f_3 < \cdots < f_{n-r}$.

**Example 5.14** (Reduced row-echelon form notation)**.** For the $5 \times 9$ matrix

$$B = \begin{bmatrix} \boxed{1} & 5 & 0 & 0 & 2 & 8 & 0 & 5 & -1 \\ 0 & 0 & \boxed{1} & 0 & 4 & 7 & 0 & 2 & 0 \\ 0 & 0 & 0 & \boxed{1} & 3 & 9 & 0 & 3 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

in reduced row-echelon form we have:

$$\begin{array}{lllll} r = 4 & & & & \\ d_1 = 1 & d_2 = 3 & d_3 = 4 & d_4 = 7 & \\ f_1 = 2 & f_2 = 5 & f_3 = 6 & f_4 = 8 & f_5 = 9 \end{array}$$

Notice that the sets

$$D = \{d_1,\, d_2,\, d_3,\, d_4\} = \{1,\, 3,\, 4,\, 7\},$$
$$F = \{2,\, 5,\, 6,\, 8,\, 9\}$$

have nothing in common and together account for all of the columns of $B$.

## 5.3 Free variables

**Definition 5.15** (Independent and Dependent Variables)**.** Suppose $A$ is the augmented matrix of a consistent system of linear equations and $B$ is a row-equivalent matrix in reduced row-echelon form. Suppose $j$ is the index of a pivot column of $B$. Then the variable $x_j$ is **dependent** . A variable that is not dependent is called **independent** or **free** .

**Example 5.16.** Describe the infinite solution set of the following system of linear equations with $m = 4$ equations in $n = 7$ variables.

$$\begin{aligned} x_1 + 4x_2 - x_4 + 7x_6 - 9x_7 &= 3 \\ 2x_1 + 8x_2 - x_3 + 3x_4 + 9x_5 - 13x_6 + 7x_7 &= 9 \\ 2x_3 - 3x_4 - 4x_5 + 12x_6 - 8x_7 &= 1 \\ -x_1 - 4x_2 + 2x_3 + 4x_4 + 8x_5 - 31x_6 + 37x_7 &= 4 \end{aligned}$$

This system has a $4 \times 8$ augmented matrix

$$\left[\begin{array}{ccccccc|c} 1 & 4 & 0 & -1 & 0 & 7 & -9 & 3 \\ 2 & 8 & -1 & 3 & 9 & -13 & 7 & 9 \\ 0 & 0 & 2 & -3 & -4 & 12 & -8 & 1 \\ -1 & -4 & 2 & 4 & 8 & -31 & 37 & 4 \end{array}\right]$$

The matrix is row-equivalent to the following matrix reduced row-echelon form (**exercise** : check this)

$$\left[\begin{array}{ccccccc|c} \boxed{1} & 4 & 0 & 0 & 2 & 1 & -3 & 4 \\ 0 & 0 & \boxed{1} & 0 & 1 & -3 & 5 & 2 \\ 0 & 0 & 0 & \boxed{1} & 2 & -6 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So we find that $r = 3$ and

$$D = \{d_1, d_2, d_3\} = \{1, 3, 4\}$$

Let $i$ denote any one of the $r = 3$ nonzero rows. Then the index $d_i$ is a pivot column. It will be easy in this case to use the equation represented by row $i$ to write an expression for the variable $x_{d_i}$. It will be a linear function of the variables $x_{f_1}, x_{f_2}, x_{f_3}, x_{f_4}$

$$\begin{aligned} &(i = 1) && x_{d_1} = x_1 = 4 - 4x_2 - 2x_5 - x_6 + 3x_7 \\ &(i = 2) && x_{d_2} = x_3 = 2 - x_5 + 3x_6 - 5x_7 \\ &(i = 3) && x_{d_3} = x_4 = 1 - 2x_5 + 6x_6 - 6x_7 \end{aligned}$$

Each element of the set $F = \{f_1,\, f_2,\, f_3,\, f_4,\, f_5\} = \{2,\, 5,\, 6,\, 7,\, 8\}$ is the index of a variable, except for $f_5 = 8$. We refer to $x_{f_1} = x_2$, $x_{f_2} = x_5$, $x_{f_3} = x_6$ and $x_{f_4} = x_7$ as **free** (or **independent**) variables since they are allowed to assume any possible combination of values that we can imagine and we can continue on to build a solution to the system by solving individual equations for the values of the other (**dependent**) variables.

Each element of the set $D = \{d_1,\, d_2,\, d_3\} = \{1,\, 3,\, 4\}$ is the index of a variable. We refer to the variables $x_{d_1} = x_1$, $x_{d_2} = x_3$ and $x_{d_3} = x_4$ as **dependent** variables since they depend on the independent variables. More precisely, for each possible choice of values for the independent variables we get exactly one set of values for the dependent variables that combine to form a solution of the system.

To express the solutions as a set, we write

$$\left\{ \begin{bmatrix} 4 - 4x_2 - 2x_5 - x_6 + 3x_7 \\ x_2 \\ 2 - x_5 + 3x_6 - 5x_7 \\ 1 - 2x_5 + 6x_6 - 6x_7 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} \middle|\; x_2,\, x_5,\, x_6,\, x_7 \in \mathbb{R} \right\} \tag{5.1}$$

or equivalently:

$$\left\{ \begin{bmatrix} 4 \\ 0 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 0 \\ -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -1 \\ 0 \\ 3 \\ 6 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_7 \begin{bmatrix} 3 \\ 0 \\ -5 \\ -6 \\ 0 \\ 0 \\ 1 \end{bmatrix} \middle|\; x_2, x_5, x_6, x_7 \in \mathbb{R} \right\}$$

The condition that $x_2,\, x_5,\, x_6,\, x_7$ are real numbers is how we specify that the variables $x_2,\, x_5,\, x_6,\, x_7$ are **free** to assume any possible values.

This systematic approach to solving a system of equations will allow us to create a precise description of the solution set for any consistent system once we have found the reduced row-echelon form of the augmented matrix. It will work just as well when the set of free variables is empty and we get just a single solution.

**Example 5.17.** Consider the system of five equations in five variables,

$$\begin{aligned} x_1 - x_2 - 2x_3 + x_4 + 11x_5 &= 13 \\ x_1 - x_2 + x_3 + x_4 + 5x_5 &= 16 \\ 2x_1 - 2x_2 + x_4 + 10x_5 &= 21 \\ 2x_1 - 2x_2 - x_3 + 3x_4 + 20x_5 &= 38 \\ 2x_1 - 2x_2 + x_3 + x_4 + 8x_5 &= 22 \end{aligned}$$

whose augmented matrix row-reduces to:

$$\left[\begin{array}{ccccc|c} \boxed{1} & -1 & 0 & 0 & 3 & 6 \\ 0 & 0 & \boxed{1} & 0 & -2 & 1 \\ 0 & 0 & 0 & \boxed{1} & 4 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{5.2}$$

Columns 1, 3 and 4 are pivot columns, so $D = \{1, 3, 4\}$. From this we know that the variables $x_1$, $x_3$ and $x_4$ will be dependent variables, and each of the $r = 3$ nonzero rows of the row-reduced matrix will yield an expression for one of these three variables. The set $F$ is all the remaining column indices, $F = \{2, 5, 6\}$. The column index $6$ in $F$ means that the final column is not a pivot column, and thus the system is consistent (see the next theorem). The remaining indices in $F$ indicate free variables, so $x_2$ and $x_5$ (the remaining variables) are our free variables. The resulting three equations that describe our solution set are then,

$$\begin{aligned} &(x_{d_1} = x_1) \qquad && x_1 = 6 + x_2 - 3x_5 \\ &(x_{d_2} = x_3) && x_3 = 1 + 2x_5 \\ &(x_{d_3} = x_4) && x_4 = 9 - 4x_5 \end{aligned}$$

Make sure you understand where these three equations came from, and notice how the location of the pivot columns determined the variables on the left-hand side of each equation. We can compactly describe the solution set as,

$$S = \left\{ \begin{bmatrix} 6 + x_2 - 3x_5 \\ x_2 \\ 1 + 2x_5 \\ 9 - 4x_5 \\ x_5 \end{bmatrix} \middle|\ x_2,\ x_5 \text{ real numbers} \right\} \tag{5.3}$$

Notice how we express the freedom for $x_2$ and $x_5$: $x_2,\ x_5$ real numbers.

**Theorem 5.18** (Recognizing Consistency of a Linear System)**.** *Suppose $A$ is the augmented matrix of a system of linear equations with $n$ variables. Suppose also that $B$ is a row-equivalent matrix in reduced row-echelon form with $r$ nonzero rows.*

- *Then the system of equations is* **inconsistent** *if and only if column $n+1$ (i.e., the last column) of $B$ is a pivot column.*
- *Equivalently a system is* **consistent** *if and only if column $n+1$ is not a pivot column of $B$.*
- *Another way of expressing the theorem is to say that a system of linear equations is* **consistent** *if and only if the last* **non-zero row** *is not $(0, 0, \ldots, 0, 1)$.*

*Proof of Recognizing Consistency of a Linear System.* (sketch, for details, see Beezer, Ver 3.5 print version p.38, proof of theorem RCLS, you can skip the proof in the textbook) If the last column vector of $B$ is a pivot column, then $B$ is in the form of:

$$\begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & * & 0 \\ & & 1 & \cdots & 0 & \cdots & * & 0 \\ & & & & 1 & \cdots & * & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 \\ & & & & & & & \end{bmatrix}$$

For the system of linear equations with the above augmented matrix, the $r+1$-st equation (i.e. the last non-zero equation) is

$$0 = 1.$$

So the system of linear equations has no solution. Conversely, if the last column vector is not a pivot column vector, then $B$ is in the form of:

$$\begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ & & 1 & \cdots & 0 & \cdots & 0 & \cdots \\ & & & & 1 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 & \cdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 \\ & & & & & & & \end{bmatrix}$$

For the system of equations with the above augmented matrix, we can move the variables corresponding to the non-pivot columns (i.e., $x_{f_1}, x_{f_2}, \ldots$) to the right hand side of the equations and therefore solve the equations. Hence it is consistent. Note that $x_{f_1}, x_{f_2}, \ldots$ are free variables. $\square$

**Example 5.19.** Determine if the following system of linear equation is consistent.

$$\begin{aligned} x_1 + x_2 + 2x_3 + 3x_4 + 2x_5 + 5x_6 &= 1 \\ 2x_1 + 2x_2 + 3x_3 - x_4 &= 1 \\ 3x_1 + 3x_2 + 5x_3 + x_4 + x_5 - 2x_6 &= 3 \\ x_4 + x_5 + 7x_6 &= 0 \end{aligned}$$

The augmented matrix is

$$\left[\begin{array}{cccccc|c} 1 & 1 & 2 & 3 & 2 & 5 & 1 \\ 2 & 2 & 3 & -1 & 0 & 0 & 1 \\ 3 & 3 & 5 & 1 & 1 & -2 & 3 \\ 0 & 0 & 0 & 1 & 1 & 7 & -1 \end{array}\right]$$

The reduced row echelon form is

$$\left[\begin{array}{cccccc|c} 1 & 1 & 0 & 0 & 5 & 62 & 0 \\ 0 & 0 & 1 & 0 & -3 & -39 & 0 \\ 0 & 0 & 0 & 1 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

The last column is a pivot column. So the system is inconsistent.

**Example 5.20.** Determine if the following system of linear equation is consistent.

$$\begin{aligned} x_1 + x_2 + 2x_3 + 3x_4 + 2x_5 + 5x_6 &= 1 \\ 2x_1 + 2x_2 + 3x_3 - x_4 &= 1 \\ 3x_1 + 3x_2 + 5x_3 + x_4 + x_5 - 2x_6 &= 3 \\ x_4 + x_5 + 7x_6 &= -1 \end{aligned}$$

The augmented matrix is

$$\left[\begin{array}{cccccc|c} 1 & 1 & 2 & 3 & 2 & 5 & 1 \\ 2 & 2 & 3 & -1 & 0 & 0 & 1 \\ 3 & 3 & 5 & 1 & 1 & -2 & 3 \\ 0 & 0 & 0 & 1 & 1 & 7 & -1 \end{array}\right]$$

The reduced row echelon form is

$$\left[\begin{array}{cccccc|c} 1 & 1 & 0 & 0 & 5 & 62 & -12 \\ 0 & 0 & 1 & 0 & -3 & -39 & 8 \\ 0 & 0 & 0 & 1 & 1 & 7 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The last column is not a pivot column. So the system is consistent.

**Theorem 5.21** (Consistent Systems r and n)**.** *Suppose $A$ is the augmented matrix of a consistent system of linear equations with $n$ variables. Suppose also that $B$ is a row-equivalent matrix in reduced row-echelon form with $r$ pivot columns. Then $r \leq n$. If $r = n$, then the system has a unique solution, and if $r < n$, then the system has infinitely many solutions.*

*Proof of Consistent Systems, $r$ and $n$.* This theorem contains three implications that we must establish. Notice first that $B$ has $n + 1$ columns, so there can be at most $n+1$ pivot columns, i.e., $r \leq n+1$. If $r = n+1$, then every column of $B$ is a pivot column, and in particular, the last column is a pivot column. So the previous theorem tells us that the system is inconsistent, contrary to our hypothesis. We are left with $r \leq n$.

When $r = n$, we find $n - r = 0$ free variables (i.e., $F = \{n+1\}$) and the only solution is given by setting the $n$ variables to the the first $n$ entries of column $n+1$ of $B$. When $r < n$, we have $n - r > 0$ free variables. Choose one free variable and set all the other free variables to zero. Now, set the chosen free variable to any fixed value. It is possible to then determine the values of the dependent variables to create a solution to the system. By setting the chosen free variable to different values, in this manner we can create infinitely many solutions. □

**Theorem 5.22** (Free Variables for Consistent Systems)**.** *Suppose $A$ is the augmented matrix of a consistent system of linear equations with $n$ variables. Suppose also that $B$ is a row-equivalent matrix in reduced row-echelon form with $r$ rows that are not completely zeros. Then the solution set can be described with $n - r$ free variables.*

**Example 5.23.** 1. System of linear equations with $n = 3$, $m = 3$.

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ 2x_1 + x_2 + x_3 &= 8 \\ x_1 + x_2 &= 5 \end{aligned}$$

Augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 1 & 1 & 8 \\ 1 & 1 & 0 & 5 \end{array}\right]$$

The reduced row echelon form of the augmented matrix.

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 3 \\ 0 & \boxed{1} & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The last column is not a pivot column. So the system of linear equations is consistent. $r = 2$, there are $3 - 2$ free variables. In fact $D = \{1, 2\}$, $F = \{3\}$. $x_1, x_2$ are dependent variables, $x_3$ is a free variables.

$$\begin{aligned} x_1 &= 3 - x_3 \\ x_2 &= 2 + x_3 \end{aligned}$$

---

2. System of linear equations with $n = 3, m = 3$.

$$\begin{aligned} -7x_1 - 6x_2 - 12x_3 &= -33 \\ 5x_1 + 5x_2 + 7x_3 &= 24 \\ x_1 + 4x_3 &= 5 \end{aligned}$$

Augmented matrix

$$\left[\begin{array}{ccc|c} -7 & -6 & -12 & -33 \\ 5 & 5 & 7 & 24 \\ 1 & 0 & 4 & 5 \end{array}\right]$$

The reduced row echelon form of the augmented matrix.

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & -3 \\ 0 & \boxed{1} & 0 & 5 \\ 0 & 0 & \boxed{1} & 2 \end{array}\right]$$

The last column is not a pivot column. So the system of linear equations is consistent. $r = 3$, there are $3 - 3 = 0$ free variables. So the solution is unique. In fact In fact

$$x_1 = -3$$
$$x_2 = 5$$
$$x_3 = 2$$

---

3. System of linear equations with $n = 2$, $m = 5$.

$$\begin{aligned} 2x_1 + 3x_2 &= 6 \\ -x_1 + 4x_2 &= -14 \\ 3x_1 + 10x_2 &= -2 \\ 3x_1 - x_2 &= 20 \\ 6x_1 + 9x_2 &= 18 \end{aligned}$$

Augmented matrix

$$\left[\begin{array}{cc|c} 2 & 3 & 6 \\ -1 & 4 & -14 \\ 3 & 10 & -2 \\ 3 & -1 & 20 \\ 6 & 9 & 18 \end{array}\right]$$

The reduced row echelon form of the augmented matrix.

$$\left[\begin{array}{cc|c} \boxed{1} & 0 & 6 \\ 0 & \boxed{1} & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The last column is not a pivot column. So the system of linear equations is consistent. $r = 2$, there are $2 - 2 = 0$ free variables. So the solution is unique. In fact

$$x_1 = 6$$
$$x_2 = -2$$

---

4. System of linear equations with $n = 4, m = 3$.

$$\begin{aligned} 2x_1 + x_2 + 7x_3 - 7x_4 &= 2 \\ -3x_1 + 4x_2 - 5x_3 - 6x_4 &= 3 \\ x_1 + x_2 + 4x_3 - 5x_4 &= 2 \end{aligned}$$

Augmented matrix

$$\left[\begin{array}{cccc|c} 2 & 1 & 7 & -7 & 2 \\ -3 & 4 & -5 & -6 & 3 \\ 1 & 1 & 4 & -5 & 2 \end{array}\right]$$

The reduced row echelon form of the augmented matrix.

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & 3 & -2 & 0 \\ 0 & \boxed{1} & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{array}\right]$$

The last column is a pivot column. Hence the system of linear equations is inconsistent. It has no solution.

---

**Theorem 5.24** (Possible Solution Sets for Linear Systems)**.** *A system of linear equations has no solutions, a unique solution or infinitely many solutions.*

*Proof of Possible Solution Sets for Linear Systems.*

- If the system is inconsistent, that it has no solutions.
- Suppose the system is consistent.
  - If it has $0$ free variable, it has a unique solution.
  - If it has $\geq 1$ free variables, it has infinite many solutions.

□

**Theorem 5.25** (Consistent More Variables than Equations Infinite solutions)**.** *Suppose a consistent system of linear equations has $m$ equations in $n$ variables. If $n > m$, then the system has infinitely many solutions.*

*Proof of Consistent, More Variables than Equations, Infinite solutions.* Suppose that the augmented matrix of the system of equations is row-equivalent to $B$, a matrix in reduced row-echelon form with $r$ nonzero rows. Because $B$ has $m$ rows in total, the number of nonzero rows is less than or equal to $m$. In other words, $r \leq m$.

Follow this with the hypothesis that $n > m$ and we find that the system has a solution set described by at least one free variable because

$$n - r \geq n - m > 0.$$

A consistent system with free variables will have an infinite number of solutions, as given by Theorem 5.21 (Consistent Systems, $r$ and $n$). $\square$

These theorems give us the procedures and implications that allow us to completely solve any system of linear equations. The main computational tool is using row operations to convert an augmented matrix into reduced row-echelon form. Here is a broad outline of how we would instruct a computer to solve a system of linear equations.

**Steps of Solving a System of Linear Equations.**

1. Represent a system of linear equations in $n$ variables by an augmented matrix.

2. Convert the matrix to a row-equivalent matrix in reduced row-echelon form using the **Gaussian Elimination** procedure given in the proof of Theorem 5.6 (Row-Equivalent Matrix in Echelon Form). Identify the location of the pivot columns, and the rank $r$.

3. If column $n + 1$ is a pivot column, then the system is inconsistent.

4. If column $n + 1$ is not a pivot column, there are two possibilities:

   (a) $r = n$ and the solution is unique. It can be read off directly from the entries in rows 1 through $n$ of column $n + 1$.

   (b) $r < n$ and there are infinitely many solutions. we can describe the solution sets by the free variables.