MATH 1030 Chapter 14

The lecture is based on Beezer, A first course in Linear algebra. Ver 3.5 Downloadable at http://linear.ups.edu/download.html .

The print version can be downloaded at http://linear.ups.edu/download/fcla-3.50-print.pdf .

14.1 Dimension

Definition 14.1 (Dimension)**.** Let V be a vector space.

Suppose a finite set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_t\}$ is a basis for V.

Then, we say that V is a **finite dimensional vector space**.

The number t (namely the number of vectors in the basis) is called the **dimension** of V.

The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be 0.

Remark. It is a non-trivial fact that the dimension is well-defined, i.e., If both $\{\mathbf{v}_1, \ldots, \mathbf{v}_t\}$ and $\{\mathbf{u}_1, \ldots, \mathbf{u}_s\}$ are bases for V, then $s = t$.

Theorem 14.2. *Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_t\}$ is a finite set of vectors which spans the vector space V. Then any set of $t + 1$ or more vectors from V is linearly dependent.*

Proof of Theorem 14.2. Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ be m vectors in V, where $m \geq t + 1$. Let $A = [\mathbf{v}_1|\mathbf{v}_2| \cdots |\mathbf{v}_t]$. Since S spans V, for every $\mathbf{u}_i$ ($1 \leq i \leq m$) there exists $\mathbf{w}_i \in \mathbb{R}^t$ such that:

$$A\mathbf{w}_i = \mathbf{u}_i.$$

Now, consider the matrix:

$$B = [\mathbf{w}_1|\mathbf{w}_2| \cdots |\mathbf{w}_m].$$

This is a $t \times m$ matrix. In particular, it has more columns than rows, due to the assumption that $m > t$.

Hence, the homogeneous linear system $\mathcal{LS}(B, \mathbf{0})$ has a non-trivial solution $\mathbf{x} \in \mathbb{R}^m$. That is:

$$B\mathbf{x} = \mathbf{0}.$$

The above equation implies that:

$$A\left(B\mathbf{x}\right) = A\mathbf{0} = \mathbf{0}.$$

By the associativity of matrix multiplication, we have:

$$A\left(B\mathbf{x}\right) = \left(AB\right)\mathbf{x}.$$

On the other hand:

$$\begin{aligned} AB &= A[\mathbf{w}_1|\mathbf{w}_2|\cdots|\mathbf{w}_m] \\ &= [A\mathbf{w}_1|A\mathbf{w}_2|\cdots|A\mathbf{w}_m] \\ &= [\mathbf{u}_1|\mathbf{u}_2|\cdots|\mathbf{u}_m] \end{aligned}$$

Hence,

$$\left(AB\right)\mathbf{x} = \mathbf{0}$$

is equivalent to:

$$[\mathbf{u}_1|\mathbf{u}_2|\cdots|\mathbf{u}_m]\mathbf{x} = \mathbf{0}$$

which is in turn equivalent to:

$$x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \cdots x_m\mathbf{u}_m = \mathbf{0}.$$

Since, $\mathbf{x}$ is not the zero vector, not all the x_i's are equal to zero. We conclude that the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ are linearly dependent. □

Theorem 14.3. *Suppose that V is a vector space with a finite basis B and a second basis C.*

Then B and C have the same size.

Proof of Theorem 14.3. Denote the size of B by t. If C has $\geq t + 1$ vectors, then by the previous theorem, C is linearly dependent, in contradiction to the condition that C is a basis.

By the same reasoning, the linearly independent set B must also not have more vectors than C.

So, B and C have the same number of vectors. □

Remark. The above theorem shows that the dimension is well-defined. No matter which basis we choose, the size is always the same.

Proposition 14.4. *Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subseteq \mathbb{R}^m$. Then*

$$\dim \operatorname{Span} S \leq n.$$

Proof of Proposition 14.4. By Theorem 13.22, there exists a subset T of S such that T is a basis for $\operatorname{Span} S$.

$$\dim \operatorname{Span} S = \text{number of vectors in } T \leq \text{number of vectors in } S = n.$$

□

Remark. Theorem 13.22 is valid if we replace $\mathbb{R}^m$ by P_n, M_{mn} or any finite dimensional vector space.

Example 14.5.

$$\dim \mathbb{R}^m = m.$$

Corollary 14.6. *Any set of n vectors in $\mathbb{R}^m$ are linearly dependent if $n > m$.*

Proof of Corollary 14.6. This follows from Theorem 14.2 and the fact that $\mathbb{R}^m$ is spanned by m vectors. □

Example 14.7. Math major only
$\dim M_{mn} = mn$. See example 3.

Example 14.8. Math major only
$\dim P_n = n + 1$. See example 4.

Example 14.9. Math major only
Let S_2 be the set of 2×2 symmetric matrices. For $A \in S_2$,

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

We can show that:

$$T = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is a basis for S_2. Hence $\dim S_2 = 3$.

Example 14.10. Math major only
Let P be the set of all real polynomials. As $\{1, x, x^2, x^3, \ldots\}$ is linearly independent, so $\dim P$ does not exists (or we can write $\dim P = \infty$).

We have seen that every column space of a matrix has a basis. Does every subspace of $\mathbb{R}^m$ have a basis?

Lemma 14.11. *Let V be a vector space and $\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{u} \in V$.*

Suppose $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent and $\mathbf{u} \notin \operatorname{Span} S$. Then $S' = \{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{u}\}$ is linearly independent.

Proof of Lemma 14.11. Let the relation of linear dependence of S' be

$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_k \mathbf{v}_k + \alpha \mathbf{u} = \mathbf{0}.$$

Suppose $\alpha \neq 0$, then

$$\mathbf{u} = -\frac{\alpha_1}{\alpha}\mathbf{v}_1 - \cdots - \frac{\alpha_k}{\alpha}\mathbf{v}_k \in \operatorname{Span} S.$$

Contradiction.

So $\alpha = 0$, then

$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_k \mathbf{v}_k = \mathbf{0}.$$

By the linear independence of S, $\alpha_i = 0$ for all i. Hence the above relation of dependence of S' is trivial. □

Theorem 14.12. *Let V be a nonzero (i.e. contains nonzero vectors) subspace of $\mathbb{R}^m$. (That is, $V \neq \{\mathbf{0}\}$.)*

Then, there exists a basis for V.

Proof of Theorem 14.12. Consider all nonempty linearly independent subsets S of vectors in V. By Corollary 14.6, the size of any such S is an integer between 1 and m.

Let n be the largest possible size of such sets, and let:

$$B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$$

be a nonempty linearly independent set of V with size n. We claim that $\operatorname{Span} B = V$:

If not, then there exists $\mathbf{u} \in V$ which does not belong to $\operatorname{Span} B$, and by Lemma 14.11 the set:

$$B \cup \mathbf{u} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n, \mathbf{u}\}$$

is an linearly indepedent set of size $n+1$, which contradicts the assumpion that n is the maximum size of linearly independent subsets in V.

Hence, the linearly independent set B spans V, and it follows that B is a basis of V. □

Alternatively,

Proof of Theorem 14.12. Let V be a nonzero vector space. Let $\mathbf{v}_1$ be a nonzero vector in V. If $V = \mathrm{Span}\ \{\mathbf{v}_1\}$, we can take $S = \{\mathbf{v}_1\}$. Then obviously $\{\mathbf{v}_1\}$ is linearly independent and hence S is a basis for V.

Otherwise, let $\mathbf{v}_2 \in V$ but not in $\mathrm{Span}\ \{\mathbf{v}_1\}$.

By the previous lemma, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent. If $V = \mathrm{Span}\ \{\mathbf{v}_1, \mathbf{v}_2\}$, we can take $S = \{\mathbf{v}_1, \mathbf{v}_2\}$.

So S is a basis for V.

Otherwise, let $\mathbf{v}_3 \in V$ but not in $\mathrm{Span}\ \{\mathbf{v}_1, \mathbf{v}_2\}$.

By the previous lemma, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. Repeat the above process, inductive we can define $\mathbf{v}_{k+1}$ as following: If $V = \mathrm{Span}\ \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, we can take $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$.

Because $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent, S is a basis for V.

Otherwise defined $\mathbf{v}_{k+1} \notin \mathrm{Span}\ \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$.

By the previous lemma, $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k+1}\}$ is linearly independent.

If the process stops, say at step k, i.e., $V = \mathrm{Span}\ \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$.

Then we can take $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$.

Because $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent, it is a basis for V.

This completes the proof.

Otherwise, the process continues infinitely, in particular, we can take $k = m + 1$ and $V \neq \mathrm{Span}\ \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{m+1}\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{m+1}\}$ is linearly independent.

Since $\langle\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}\rangle = \mathbb{R}^m$, by Theorem 14.2 the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{m+1}\}$ are linearly dependent. Contradiction. □

Theorem 14.13. *Suppose a vector space V has dimension n. Then, any linearly independent set with n vectors in V is a basis for V.*

Theorem 14.14. *Suppose a vector space V has dimension n. Suppose S is a set of n vectors in V which spans V (That is, $\langle S\rangle = V$).*

Then, S is a basis for V.

14.2 Rank and nullity of a matrix

Definition 14.15 (Nullity of a matrix)**.** Suppose that $A \in M_{mn}$. Then the **nullity** of A is the dimension of the null space of A, $n(A) = \dim(\mathcal{N}(A))$.

Definition 14.16 (Rank of a matrix)**.** Suppose that $A \in M_{mn}$. Then the **rank** of A is the dimension of the column space of A, $r(A) = \dim(\mathcal{C}(A))$.

Example 14.17. Rank and nullity of a matrix

Let us compute the rank and nullity of

$$A = \begin{bmatrix} 2 & -4 & -1 & 3 & 2 & 1 & -4 \\ 1 & -2 & 0 & 0 & 4 & 0 & 1 \\ -2 & 4 & 1 & 0 & -5 & -4 & -8 \\ 1 & -2 & 1 & 1 & 6 & 1 & -3 \\ 2 & -4 & -1 & 1 & 4 & -2 & -1 \\ -1 & 2 & 3 & -1 & 6 & 3 & -1 \end{bmatrix}$$

To do this, we will first row-reduce the matrix since that will help us determine bases for the null space and column space.

$$\begin{bmatrix} \boxed{1} & -2 & 0 & 0 & 4 & 0 & 1 \\ 0 & 0 & \boxed{1} & 0 & 3 & 0 & -2 \\ 0 & 0 & 0 & \boxed{1} & -1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From this row-equivalent matrix in reduced row-echelon form we record $D = \{1, 3, 4, 6\}$ and $F = \{2, 5, 7\}$.

By Theorem 13.10 (Basis of the Column Space), for each index in D, we can create a single basis vector. In fact $T = \{\mathbf{A}_1, \mathbf{A}_3, \mathbf{A}_4, \mathbf{A}_6\}$ is a basis for $\mathcal{C}(A)$. In total the basis will have 4 vectors, so the column space of A will have dimension 4 and we write $r(A) = 4$.

By Theorem 11.12, for each index in F, we can create a single basis vector. In total the basis will have 3 vectors, so the null space of A will have dimension 3 and we write $n(A) = 3$. In fact:

$$R = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -3 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ 3 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

is a basis for $\mathcal{N}(A)$.

Theorem 14.18 (Computing rank and nullity)**.** *Suppose $A \in M_{mn}$ and $A \xrightarrow{RREF} B$. Let r denote the number of pivot columns (= number of nonzero rows). Then $r(A) = r$ and $n(A) = n - r$.*

Proof of Computing rank and nullity. Let $D = \{d_1, \ldots, d_r\}$ be the indexes of the pivot columns of B. By Theorem 13.10 (Basis of the Column Space), $\{\mathbf{A}_{d_1}, \ldots, \mathbf{A}_{d_r}\}$ is a basis for $\mathcal{C}(A)$. So $r(A) = r$.

By Theorem 11.12, each free variable corresponding to a single basis vector for the null space. So $n(A)$ is the number of free variables $= n - r$. □

Corollary 14.19 (Dimension Formula)**.** *Suppose $A \in M_{mn}$, then*

$$r(A) + n(A) = n.$$

Theorem 14.20. *Let A be a $m \times n$ matrix. Then*

$$r(A) = r\left(A^t\right).$$

Equivalently

$$\dim \mathcal{C}(A) = \dim \mathcal{R}(A).$$

Proof of Theorem 14.20. Let $A \xrightarrow{\text{RREF}} B$.

Let r denote the number of pivot columns (= number of nonzero rows).

Then by the above discussion $r = r(A)$. By Theorem 13.19 (Basis for the Row Space), the first r columns of B^t form a basis for $\mathcal{R}(A) = \mathcal{C}(A^t)$. Hence $r = r(A^t)$. This completes the proof. □

Let us take a look at the rank and nullity of a square matrix.

Example 14.21. The matrix

$$E = \begin{bmatrix} 0 & 4 & -1 & 2 & 2 & 3 & 1 \\ 2 & -2 & 1 & -1 & 0 & -4 & -3 \\ -2 & -3 & 9 & -3 & 9 & -1 & 9 \\ -3 & -4 & 9 & 4 & -1 & 6 & -2 \\ -3 & -4 & 6 & -2 & 5 & 9 & -4 \\ 9 & -3 & 8 & -2 & -4 & 2 & 4 \\ 8 & 2 & 2 & 9 & 3 & 0 & 9 \end{bmatrix}$$

is row-equivalent to the matrix in reduced row-echelon form,

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

With $n = 7$ columns and $r = 7$ nonzero rows tells us the rank is $r(E) = 7$ and the nullity is $n(E) = 7 - 7 = 0$.

The value of either the nullity or the rank are enough to characterize a nonsingular matrix.

Theorem 14.22 (Rank and Nullity of a Nonsingular Matrix)**.** *Suppose that A is a square matrix of size n. The following are equivalent.*

1. *A is nonsingular.*

2. *The rank of A is n, $r(A) = n$.*

3. *The nullity of A is zero, $n(A) = 0$.*

Proof of Rank and Nullity of a Nonsingular Matrix. ($1 \Rightarrow 2$) If A is nonsingular then $\mathcal{C}(A) = \mathbb{R}^n$.

If $\mathcal{C}(A) = \mathbb{R}^n$, then the column space has dimension n, so the rank of A is n.

($2 \Rightarrow 3$) Suppose $r(A) = n$. Then the dimension formula gives

$$\begin{aligned} n(A) &= n - r(A) \\ &= n - n \\ &= 0 \end{aligned}$$

($3 \Rightarrow 1$) Suppose $n(A) = 0$, so a basis for the null space of A is the empty set. This implies that $\mathcal{N}(A) = \{\mathbf{0}\}$ and hence A is nonsingular. □

With a new equivalence for a nonsingular matrix, we can update our list of equivalences which now becomes a list requiring double digits to number.

Theorem 14.23. *Suppose that A is a square matrix of size n. The following are equivalent.*

1. *A is nonsingular.*

2. *A row-reduces to the identity matrix.*
3. *The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.*
4. *The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.*
5. *The columns of A are a linearly independent set.*
6. *A is invertible.*
7. *The column space of A is $\mathbb{R}^n$, $\mathcal{C}(A) = \mathbb{R}^n$.*
8. *The columns of A are a basis for $\mathbb{R}^n$.*
9. *The rank of A is n, $r(A) = n$.*
10. *The nullity of A is zero, $n(A) = 0$.*