MATH 1030 Chapter 11

The lecture is based on Beezer, A first course in Linear algebra. Ver 3.5 Downloadable at http://linear.ups.edu/download.html .

The print version can be downloaded at http://linear.ups.edu/download/fcla-3.50-print.pdf .

Reference.

- Beezer, Ver 3.5 Section SS (print version p83 - p94)
- Strang, Sect 2.3

Exercise.

- Exercises with solutions can be downloaded at http://linear.ups.edu/download/fcla-3.50-solution-manual.pdf Section SS (p.34-40) C40-45, C50, C60, M10, M11, M12. (Replace $\mathbb{C}$ by $\mathbb{R}$ in the following questions) T10, T20, T21, T22.
- Strang, Sect 2.3.

In this section we will provide an extremely compact way to describe an infinite set of vectors, making use of linear combinations. This will give us a convenient way to describe the solution set of a linear system, the null space of a matrix, and many other sets of vectors.

11.1 Span of a Set of Vectors

Definition 11.1 (Span of a Set of Column Vectors)**.** Given a set of vectors

$$S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\},$$

their **span**, $\operatorname{Span} S$, is the set of all linear combinations of $\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3,\ \ldots,\ \mathbf{u}_n$. Symbolically,

$$\begin{aligned}\operatorname{Span} S &= \{\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_n\mathbf{u}_n \mid \alpha_i \in \mathbb{R},\ 1 \le i \le n\} \\ &= \left\{ \sum_{i=1}^{n} \alpha_i\mathbf{u}_i \,\middle|\, \alpha_i \in \mathbb{R},\ 1 \le i \le n \right\}\end{aligned}$$

Theorem 11.2. *Let* $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\} \subseteq V = \mathbb{R}^m$. *Then* $\operatorname{Span} S$ *is a subspace of* V.

Proof of Theorem 11.2. Obviously $\operatorname{Span} S$ is nonempty. Let $\alpha \in \mathbb{R}$, $\mathbf{v}, \mathbf{w} \in W = \operatorname{Span} S$. Then there exists $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k$ such that

$$\mathbf{v} = \alpha_1\mathbf{u}_1 + \cdots + \alpha_k\mathbf{u}_k,$$

$$\mathbf{w} = \beta_1\mathbf{u}_1 + \cdots + \beta_k\mathbf{u}_k.$$

Then

$$\alpha\mathbf{v} + \mathbf{w} = (\alpha\alpha_1 + \beta_1)\mathbf{u}_1 + \cdots + (\alpha\alpha_k + \beta_k)\mathbf{u}_k$$

is in $\operatorname{Span} S$. Thus by Theorem 9.23, W is a subspace. $\square$

Main Questions.

1. Determine if a vector $\mathbf{v}$ is an element of $\operatorname{Span} S$.
2. Describe the set $\operatorname{Span} S$.
3. Is $\operatorname{Span} S$ equal to $\mathbb{R}^m$?

Example 11.3. Consider the following set of 5 vectors, S, from $\mathbb{R}^4$:

$$S = \left\{ \begin{bmatrix}1\\1\\3\\1\end{bmatrix}, \begin{bmatrix}2\\1\\2\\-1\end{bmatrix}, \begin{bmatrix}7\\3\\5\\-5\end{bmatrix}, \begin{bmatrix}1\\1\\-1\\2\end{bmatrix}, \begin{bmatrix}-1\\0\\9\\0\end{bmatrix} \right\}.$$

Consider the infinite set of vectors $\operatorname{Span} S$ formed by all linear combinations of the elements of S. Here are four vectors which we definitely know are elements of $\operatorname{Span} S$:

$$\mathbf{w} = (2)\begin{bmatrix}1\\1\\3\\1\end{bmatrix} + (1)\begin{bmatrix}2\\1\\2\\-1\end{bmatrix} + (-1)\begin{bmatrix}7\\3\\5\\-5\end{bmatrix} + (2)\begin{bmatrix}1\\1\\-1\\2\end{bmatrix} + (3)\begin{bmatrix}-1\\0\\9\\0\end{bmatrix} = \begin{bmatrix}-4\\2\\28\\10\end{bmatrix}$$

$$\mathbf{x} = (5)\begin{bmatrix}1\\1\\3\\1\end{bmatrix} + (-6)\begin{bmatrix}2\\1\\2\\-1\end{bmatrix} + (-3)\begin{bmatrix}7\\3\\5\\-5\end{bmatrix} + (4)\begin{bmatrix}1\\1\\-1\\2\end{bmatrix} + (2)\begin{bmatrix}-1\\0\\9\\0\end{bmatrix} = \begin{bmatrix}-26\\-6\\2\\34\end{bmatrix}$$

$$\mathbf{y} = (1)\begin{bmatrix}1\\1\\3\\1\end{bmatrix} + (0)\begin{bmatrix}2\\1\\2\\-1\end{bmatrix} + (1)\begin{bmatrix}7\\3\\5\\-5\end{bmatrix} + (0)\begin{bmatrix}1\\1\\-1\\2\end{bmatrix} + (1)\begin{bmatrix}-1\\0\\9\\0\end{bmatrix} = \begin{bmatrix}7\\4\\17\\-4\end{bmatrix}$$

$$\mathbf{z} = (0)\begin{bmatrix}1\\1\\3\\1\end{bmatrix} + (0)\begin{bmatrix}2\\1\\2\\-1\end{bmatrix} + (0)\begin{bmatrix}7\\3\\5\\-5\end{bmatrix} + (0)\begin{bmatrix}1\\1\\-1\\2\end{bmatrix} + (0)\begin{bmatrix}-1\\0\\9\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}.$$

Fundamental question: Determine if a given vector is an element of the set or not. Let us learn more about $\operatorname{Span} S$ by investigating which vectors are elements of the set, and which are not.

First, is $\mathbf{u} = \begin{bmatrix}-15\\-6\\19\\5\end{bmatrix}$ an element of $\operatorname{Span} S$?

In other words, are there scalars $\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5$ such that:

$$\alpha_1\begin{bmatrix}1\\1\\3\\1\end{bmatrix} + \alpha_2\begin{bmatrix}2\\1\\2\\-1\end{bmatrix} + \alpha_3\begin{bmatrix}7\\3\\5\\-5\end{bmatrix} + \alpha_4\begin{bmatrix}1\\1\\-1\\2\end{bmatrix} + \alpha_5\begin{bmatrix}-1\\0\\9\\0\end{bmatrix} = \mathbf{u} = \begin{bmatrix}-15\\-6\\19\\5\end{bmatrix}?$$

Searching for such scalars is equivalent to finding a solution to the linear system of equations with augmented matrix:

$$\left[\begin{array}{ccccc|c} 1 & 2 & 7 & 1 & -1 & -15\\ 1 & 1 & 3 & 1 & 0 & -6\\ 3 & 2 & 5 & -1 & 9 & 19\\ 1 & -1 & -5 & 2 & 0 & 5 \end{array}\right].$$

This matrix row-reduces to

$$\left[\begin{array}{ccccc|c} \boxed{1} & 0 & -1 & 0 & 3 & 10\\ 0 & \boxed{1} & 4 & 0 & -1 & -9\\ 0 & 0 & 0 & \boxed{1} & -2 & -7\\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

At this point, we see that the system is consistent, so we know there is a solution for the five scalars $\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5$. This is enough evidence for us to say that $\mathbf{u} \in \operatorname{Span} S$.

Moreover, we can compute an actual solution, for example:

$$\alpha_1 = 2 \qquad \alpha_2 = 1 \qquad \alpha_3 = -2 \qquad \alpha_4 = -3 \qquad \alpha_5 = 2.$$

This particular solution allows us to write

$$(2)\begin{bmatrix}1\\1\\3\\1\end{bmatrix} + (1)\begin{bmatrix}2\\1\\2\\-1\end{bmatrix} + (-2)\begin{bmatrix}7\\3\\5\\-5\end{bmatrix} + (-3)\begin{bmatrix}1\\1\\-1\\2\end{bmatrix} + (2)\begin{bmatrix}-1\\0\\9\\0\end{bmatrix} = \mathbf{u} = \begin{bmatrix}-15\\-6\\19\\5\end{bmatrix}$$

making it even more obvious that $\mathbf{u} \in \operatorname{Span} S$.

We now determine if $\mathbf{v} = \begin{bmatrix}3\\1\\2\\-1\end{bmatrix}$ an element of $\operatorname{Span} S$.

We want to know if there are scalars $\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5$ such that:

$$\alpha_1\begin{bmatrix}1\\1\\3\\1\end{bmatrix} + \alpha_2\begin{bmatrix}2\\1\\2\\-1\end{bmatrix} + \alpha_3\begin{bmatrix}7\\3\\5\\-5\end{bmatrix} + \alpha_4\begin{bmatrix}1\\1\\-1\\2\end{bmatrix} + \alpha_5\begin{bmatrix}-1\\0\\9\\0\end{bmatrix} = \mathbf{v} = \begin{bmatrix}3\\1\\2\\-1\end{bmatrix}$$

Again, this is equivalent to finding a solution to the linear system of equations with augmented matrix:

$$\left[\begin{array}{ccccc|c} 1 & 2 & 7 & 1 & -1 & 3\\ 1 & 1 & 3 & 1 & 0 & 1\\ 3 & 2 & 5 & -1 & 9 & 2\\ 1 & -1 & -5 & 2 & 0 & -1 \end{array}\right].$$

This matrix row-reduces to

$$\left[\begin{array}{ccccc|c} \boxed{1} & 0 & -1 & 0 & 3 & 0\\ 0 & \boxed{1} & 4 & 0 & -1 & 0\\ 0 & 0 & 0 & \boxed{1} & -2 & 0\\ 0 & 0 & 0 & 0 & 0 & \boxed{1} \end{array}\right].$$

At this point, we see that the system is inconsistent, so we know there is no solution for the five scalars $\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5$. This is enough evidence for us to say that $\mathbf{v} \notin \operatorname{Span} S$. End of story.

From the previous example, we have the following theorem:

Theorem 11.4. *Suppose that* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$ *are in* $\mathbf{R}^m$. *Let* A *be the* $m \times n$ *matrix whose* i*-th column is* $\mathbf{u}_i$. *Then* $\mathbf{v} \in \operatorname{Span} S$ *if and only if* $\mathcal{LS}(A, \mathbf{v})$ *is consistent.*

Example 11.5. Computational Technique.

Given $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ in $\mathbb{R}^m$, determine if

$$\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} \in \operatorname{Span} S.$$

Solution.

1. To determine if $\mathbf{v} \in \operatorname{Span} S$, we need to find $\alpha_1, \ldots, \alpha_n$ such that

$$\alpha_1 \mathbf{u}_1 + \cdots + \alpha_n \mathbf{u}_n = \mathbf{v}.$$

2. This is equivalent to solving the system of linear equations $\mathcal{LS}(A, \mathbf{v})$, where A is the $m \times n$ matrix whose i-th column is $\mathbf{u}_i$.

3. Row-reduce the augmented matrix $[A|\mathbf{v}]$ to an RREF B.

   (a) If the last column of B is a pivot column, then the system is inconsistent and $\mathbf{v} \notin \operatorname{Span} S$.

   (b) If the last column of B is not a pivot column, then the system is consistent and $\mathbf{v} \in \operatorname{Span} S$.

Example 11.6. For the set S defined as in Example 11.3, determine if:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_4 \end{bmatrix} \in \operatorname{Span} S.$$

Applying Gauss-Jordan elimination to the augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 2 & 7 & 1 & -1 & v_1 \\ 1 & 1 & 3 & 1 & 0 & v_2 \\ 3 & 2 & 5 & -1 & 9 & v_3 \\ 1 & -1 & -5 & 2 & 0 & v_4 \end{array}\right],$$

we obtain

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 3 & -3v_1 + 5v_2 - v_4 \\ 0 & 1 & 4 & 0 & -1 & v_1 - v_2 \\ 0 & 0 & 0 & 1 & -2 & 2v_1 - 3v_2 + v_4 \\ 0 & 0 & 0 & 0 & 0 & 9v_1 - 16v_2 + v_3 + 4v_4 \end{array}\right].$$

If $9v_1 - 16v_2 + v_3 + 4v_4 = 0$, then the last column is not a pivot column and the above is an RREF. In this case, $\mathbf{v} \in \operatorname{Span} S$.

If instead $9v_1 - 16v_2 + v_3 + 4v_4 \neq 0$, then the above matrix is not an RREF. Hence, the corresponding system of linear equations is inconsistent and thus $\mathbf{v} \notin \operatorname{Span} S$.

We therefore conclude that $\mathbf{v} \in \operatorname{Span} S$ if and only if:

$$9v_1 - 16v_2 + v_3 + 4v_4 = 0.$$

Example 11.7. Consider:

$$S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},\, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix},\, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

and consider the infinite set $\operatorname{Span} S$.

Does $\mathbf{w} = \begin{bmatrix} 1 \\ 8 \\ 5 \end{bmatrix}$ lie in $\operatorname{Span} S$?

To answer this question, we will look for scalars $\alpha_1,\, \alpha_2,\, \alpha_3$ such that

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 = \mathbf{w}.$$

This is equivalent to solving the system of linear equations

$$\begin{aligned} \alpha_1 - \alpha_2 + 2\alpha_3 &= 1 \\ 2\alpha_1 + \alpha_2 + \alpha_3 &= 8 \\ \alpha_1 + \alpha_2 &= 5. \end{aligned}$$

Building the augmented matrix for this linear system, and row-reducing, gives:

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 3 \\ 0 & \boxed{1} & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

This system has infinitely many solutions (there is a free variable in x_3), but all we need is one solution vector. The solution,

$$\alpha_1 = 2 \qquad \alpha_2 = 3 \qquad \alpha_3 = 1$$

tells us that:

$$(2)\mathbf{u}_1 + (3)\mathbf{u}_2 + (1)\mathbf{u}_3 = \mathbf{w}.$$

So we are convinced that $\mathbf{w}$ really is in $\operatorname{Span} S$.

Notice that there is an infinite number of ways to answer this question affirmatively. We could choose a different solution, this time choosing the free variable to be zero,

$$\alpha_1 = 3 \qquad \alpha_2 = 2 \qquad \alpha_3 = 0,$$

showing that

$$(3)\mathbf{u}_1 + (2)\mathbf{u}_2 + (0)\mathbf{u}_3 = \mathbf{w}.$$

Verifying the arithmetic in this second solution will make it obvious that $\mathbf{w}$ is in this span. And of course, we now realize that there are an *infinite* number of ways to realize $\mathbf{w}$ as element of $\operatorname{Span} S$.

Let us ask the same type of question again, but this time with: $\mathbf{y} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$.

Is $\mathbf{y} \in \operatorname{Span} S$?

So we will look for scalars $\alpha_1, \alpha_2, \alpha_3$ such that

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 = \mathbf{y}.$$

This is equivalent to finding solutions to the system of equations

$$\begin{aligned} \alpha_1 - \alpha_2 + 2\alpha_3 &= 2 \\ 2\alpha_1 + \alpha_2 + \alpha_3 &= 4 \\ \alpha_1 + \alpha_2 &= 3, \end{aligned}$$

Building the augmented matrix for this linear system and row-reducing gives

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 0 \\ 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{array}\right].$$

This system is inconsistent because the last column is a pivot column. So there are no scalars $\alpha_1, \alpha_2, \alpha_3$ that will create a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ that equals $\mathbf{y}$. More precisely, $\mathbf{y} \notin \operatorname{Span} S$.

There are three things to observe in this example.

1. It is easy to construct vectors in $\operatorname{Span} S$.
2. It is possible that some vectors are in $\operatorname{Span} S$ (such as $\mathbf{w}$), while others are not (such as $\mathbf{y}$).
3. Deciding if a given vector is in $\operatorname{Span} S$ leads to a linear system of equations and asking if the system is consistent.

Example 11.8. Let

$$R = \{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3\} = \left\{ \begin{bmatrix} -7 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} -12 \\ 7 \\ 4 \end{bmatrix} \right\}$$

and consider its span $\operatorname{Span} R$.

Does the vector $\mathbf{z} = \begin{bmatrix} -33 \\ 24 \\ 5 \end{bmatrix}$ lie in $\operatorname{Span} R$?

To answer this question, we will look for scalars $\alpha_1,\, \alpha_2,\, \alpha_3$ so that

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 = \mathbf{z}.$$

This is equivalent to finding solutions to the following system of linear equations:

$$\begin{aligned} -7\alpha_1 - 6\alpha_2 - 12\alpha_3 &= -33 \\ 5\alpha_1 + 5\alpha_2 + 7\alpha_3 &= 24 \\ \alpha_1 + 4\alpha_3 &= 5. \end{aligned}$$

Building the augmented matrix for this linear system and row-reducing gives

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & -3 \\ 0 & \boxed{1} & 0 & 5 \\ 0 & 0 & \boxed{1} & 2 \end{array}\right].$$

This system has a unique solution,

$$\alpha_1 = -3 \quad \alpha_2 = 5 \quad \alpha_3 = 2$$

telling us that

$$(-3)\mathbf{v}_1 + (5)\mathbf{v}_2 + (2)\mathbf{v}_3 = \mathbf{z}.$$

So we are convinced that $\mathbf{z}$ really is in $\operatorname{Span} R$. Notice that in this case we have only one way to answer the question affirmatively, since the solution is unique.

Let us ask about another vector. Let $\mathbf{x} = \begin{bmatrix} -7 \\ 8 \\ -3 \end{bmatrix}$. Is $\mathbf{x}$ a vector in $\operatorname{Span} R$?

In other words, are there scalars $\alpha_1,\ \alpha_2,\ \alpha_3$ so that:

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{x}$$

This is equivalent to finding the solutions to the system of equations

$$\begin{aligned} -7\alpha_1 - 6\alpha_2 - 12\alpha_3 &= -7 \\ 5\alpha_1 + 5\alpha_2 + 7\alpha_3 &= 8 \\ \alpha_1 + 4\alpha_3 &= -3. \end{aligned}$$

Building the augmented matrix for this linear system and row-reducing gives

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & \boxed{1} & -1 \end{array}\right].$$

This system has a unique solution,

$$\alpha_1 = 1 \quad \alpha_2 = 2 \quad \alpha_3 = -1$$

telling us that

$$(1)\mathbf{v}_1 + (2)\mathbf{v}_2 + (-1)\mathbf{v}_3 = \mathbf{x}.$$

So we are convinced that $\mathbf{x}$ really is in $\operatorname{Span} R$. Notice that in this case we again have only one way to answer the question affirmatively since the solution is again unique.

We could continue to test other vectors for membership in $\operatorname{Span} R$, but there is no point. A question about membership in $\operatorname{Span} R$ inevitably leads to a system of three equations in the three variables $\alpha_1,\ \alpha_2,\ \alpha_3$ with a coefficient matrix whose columns are the vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3$. This particular coefficient matrix is nonsingular, so by Theorem 7.25 (Nonsingular Matrix Equivalences) the system is guaranteed to have a solution. (This solution is unique, but that is not critical here.) So no matter which vector we might have chosen for $\mathbf{z}$, we are certain to discover that it was an element of $\operatorname{Span} R$.

Conclusion: Every vector of size 3 is in $\operatorname{Span} R$, or $\operatorname{Span} R = \mathbb{R}^3$.

The previous example above inspires the following result:

Theorem 11.9. *Given m vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_m\}$ in $\mathbb{R}^m$, let A be the $m \times m$ square matrix whose i-th column is $\mathbf{u}_i$. Then A is non-singular if and only if* $\operatorname{Span} S = \mathbb{R}^m$.

Proof of Theorem 11.9. ($\Rightarrow$) If A is non-singular, then for every $\mathbf{b} \in \mathbb{R}^m$ the equation $\mathcal{LS}(A, \mathbf{b})$ is consistent by Theorem 7.24 (Nonsingular Matrices and Unique Solutions). Hence $\mathbf{b} \in \operatorname{Span} S$. So $\mathbb{R}^m = \operatorname{Span} S$.

($\Leftarrow$) This part is difficult; we will only sketch the idea. Let the RREF of A be B. Suppose that A is singular. Then $B \neq I_m$ and the last row of B is a zero row. So there exists $\mathbf{b} \in \mathbb{R}^m$ such that $\mathcal{LS}(B, \mathbf{b})$ is inconsistent. (e.g. $\mathbf{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$).

Hence there exists $\mathbf{c} \in \mathbb{R}^m$ such that $\mathcal{LS}(A, \mathbf{c})$ is inconsistent (why?). Thus $\mathbf{c} \notin \operatorname{Span} S$. So $\operatorname{Span} S \neq \mathbb{R}^m$. This completes the proof.

Alternatively: Suppose A is singular. We claim that the span of S is not equal to $\mathbb{R}^m$:

If A is singular, then there exists a sequence of elementary matrices J_i such that: $J_l J_{l-1} \cdots J_2 J_1 A$ is an RREF matrix B whose last row is a zero row.

Let $J = J_l J_{l-1} \cdots J_2 J_1$, which is invertible. We claim that the vector:

$$\vec{v} = J^{-1}\vec{e}_m = J^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

(which incidentally is the last column of the matrix J^{-1}) does not lie in $\operatorname{Span} S$.

Suppose $\vec{v} \in \operatorname{Span} S$, then exists a vector $\vec{x} \in \mathbb{R}^m$ such that:

$$A\vec{x} = \vec{v} = J^{-1}\vec{e}_m$$

Multiplying both sides with J from the left, we have:

$$JA\vec{x} = \vec{e}_m.$$

But $JA = B$, whose last row is the zero row, which implies that the last component of the vector $JA\vec{x}$ is equal to zero, contradicting the fact that the last component of $\vec{e}_m$ is equal to 1.

It follows that if A is singular, then $\operatorname{Span} S \neq \mathbb{R}^m$. $\square$

11.2 Obtain Same Span Using Fewer Vectors

Example 11.10. Begin with the following set of four vectors of size 3:

$$T = \{\mathbf{w}_1,\, \mathbf{w}_2,\, \mathbf{w}_3,\, \mathbf{w}_4\} = \left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ -5 \\ 4 \end{bmatrix}, \begin{bmatrix} -7 \\ -6 \\ -5 \end{bmatrix} \right\}.$$

Let:

$$D = \begin{bmatrix} 2 & 1 & 7 & -7 \\ -3 & 4 & -5 & -6 \\ 1 & 1 & 4 & -5 \end{bmatrix}$$

and consider the infinite set $W = \operatorname{Span} T$. Check that the vector

$$\mathbf{z}_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

is a solution to the homogeneous system $\mathcal{LS}(D, \mathbf{0})$

We can write the linear combination,

$$2\mathbf{w}_1 + 3\mathbf{w}_2 + 0\mathbf{w}_3 + 1\mathbf{w}_4 = \mathbf{0}$$

which we can solve for $\mathbf{w}_4$ as

$$\mathbf{w}_4 = (-2)\mathbf{w}_1 + (-3)\mathbf{w}_2.$$

This equation says that whenever we encounter the vector $\mathbf{w}_4$, we can replace it with a specific linear combination of the vectors $\mathbf{w}_1$ and $\mathbf{w}_2$. So using $\mathbf{w}_4$ in the set T, along with $\mathbf{w}_1$ and $\mathbf{w}_2$, is excessive. An example of what we mean here can be illustrated by the computation:

$$\begin{aligned} &5\mathbf{w}_1 + (-4)\mathbf{w}_2 + 6\mathbf{w}_3 + (-3)\mathbf{w}_4 \\ &\quad = 5\mathbf{w}_1 + (-4)\mathbf{w}_2 + 6\mathbf{w}_3 + (-3)\left((-2)\mathbf{w}_1 + (-3)\mathbf{w}_2\right) \\ &\quad = 5\mathbf{w}_1 + (-4)\mathbf{w}_2 + 6\mathbf{w}_3 + (6\mathbf{w}_1 + 9\mathbf{w}_2) \\ &\quad = 11\mathbf{w}_1 + 5\mathbf{w}_2 + 6\mathbf{w}_3. \end{aligned}$$

So, what began as a linear combination of the vectors $\mathbf{w}_1,\, \mathbf{w}_2,\, \mathbf{w}_3,\, \mathbf{w}_4$ has been reduced to a linear combination of the vectors $\mathbf{w}_1,\, \mathbf{w}_2,\, \mathbf{w}_3$.

Hence:

$$W = \text{Span}\{\mathbf{w}_1,\ \mathbf{w}_2,\ \mathbf{w}_3\},$$

and the span of our set of vectors, W, has not changed, but we have described it by the span of a set of three vectors, rather than four. Furthermore, we can achieve yet another, similar, reduction.

Check that the vector

$$\mathbf{z}_1 = \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

is a solution to the homogeneous system $\mathcal{LS}(D, \mathbf{0})$.

We can write the linear combination,

$$(-3)\mathbf{w}_1 + (-1)\mathbf{w}_2 + 1\mathbf{w}_3 = \mathbf{0}$$

which we can solve for $\mathbf{w}_3$ as

$$\mathbf{w}_3 = 3\mathbf{w}_1 + 1\mathbf{w}_2.$$

This equation says that whenever we encounter the vector $\mathbf{w}_3$, we can replace it with a specific linear combination of the vectors $\mathbf{w}_1$ and $\mathbf{w}_2$. So, as before, the vector $\mathbf{w}_3$ is not needed in the description of W, provided we have $\mathbf{w}_1$ and $\mathbf{w}_2$ available. In particular, a careful proof would show that

$$W = \text{Span}\{\mathbf{w}_1,\ \mathbf{w}_2\}$$

So W began life as the span of a set of four vectors. We have now shown (utilizing solutions to a homogeneous system) that W can also be described as the span of a set of just two vectors. Convince yourself that we cannot go any further. In other words, it is not possible to dismiss either $\mathbf{w}_1$ or $\mathbf{w}_2$ in a similar fashion and winnow the set down to just one vector.

What was it about the original set of four vectors that allowed us to declare certain vectors as surplus? And just which vectors were we able to dismiss? And why did we have to stop once we had two vectors remaining? The answers to these questions motivate **linear independence**, our next section and next definition, and so are worth considering carefully now.

In general:

Theorem 11.11. *Let* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *be a set of vectors in a vector space.*
If $\mathbf{w}$ *lies in* $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$*, then:*

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n, \mathbf{w}\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$$

11.3 Geometric Interpretation

Open in browser

11.4 Spanning Sets of Null Spaces

Recall Theorem 9.26

Theorem 11.12. *Spanning Sets for Null Spaces Suppose that A is an $m\times n$ matrix and B is a row-equivalent matrix in reduced row-echelon form. Suppose that B has r pivot columns, with indices given by $D = \{d_1, d_2, d_3, \ldots, d_r\}$, while the $n-r$ non-pivot columns have indices $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$. Construct the $n-r$ vectors $\mathbf{z}_j$, $1 \le j \le n-r$ of size n,*

$$[\mathbf{z}_j]_i = \begin{cases} 1 & \text{if } i \in F,\ i = f_j \\ 0 & \text{if } i \in F,\ i \neq f_j \\ -[B]_{k,f_j} & \text{if } i \in D,\ i = d_k \end{cases}$$

Then the null space of A is given by

$$\mathcal{N}(A) = \operatorname{Span}\{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \ldots, \mathbf{z}_{n-r}\}$$

Proof of Theorem 11.12. The can be seen by moving the free variables to another side. For details. See Beezer p88. Don't memorize this theorem. Instead, study the examples below. □

Example 11.13. Spanning set of a null space

Find a set of vectors, S, so that the null space of the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 & -1 & -5 \\ 2 & 5 & 7 & 1 & 1 \\ 1 & 1 & 5 & 1 & 5 \\ -1 & -4 & -2 & 0 & 4 \end{bmatrix}$$

is the span of S, that is, $\operatorname{Span} S = \mathcal{N}(A)$.

The null space of A is the set of all solutions to the homogeneous system $\mathcal{LS}(A, \mathbf{0})$. Begin by row-reducing A. The result is

$$\begin{bmatrix} \boxed{1} & 0 & 6 & 0 & 4 \\ 0 & \boxed{1} & -1 & 0 & -2 \\ 0 & 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We have $D = \{1, 2, 4\}$ and $F = \{3, 5\}$. Hence x_3 and x_5 are free variables and we can interpret each nonzero row as an expression for the dependent variables x_1, x_2, x_4 (respectively) in the free variables x_3 and x_5. With this we can write the vector form of a solution vector as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -6x_3 - 4x_5 \\ x_3 + 2x_5 \\ x_3 \\ -3x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -6 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -4 \\ 2 \\ 0 \\ -3 \\ 1 \end{bmatrix}.$$

Then, in the notation of the above theorem, we have

$$\mathbf{z}_1 = \begin{bmatrix} -6 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{z}_2 = \begin{bmatrix} -4 \\ 2 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

and

$$\mathcal{N}(A) = \text{Span}\,\{\mathbf{z}_1, \mathbf{z}_2\} = \text{Span}\left\{ \begin{bmatrix} -6 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}.$$

Example 11.14. Consider the matrix:

$$A = \begin{bmatrix} 2 & 1 & 5 & 1 & 5 & 1 \\ 1 & 1 & 3 & 1 & 6 & -1 \\ -1 & 1 & -1 & 0 & 4 & -3 \\ -3 & 2 & -4 & -4 & -7 & 0 \\ 3 & -1 & 5 & 2 & 2 & 3 \end{bmatrix}.$$

Row-reducing A gives the matrix

$$B = \begin{bmatrix} \boxed{1} & 0 & 2 & 0 & -1 & 2 \\ 0 & \boxed{1} & 1 & 0 & 3 & -1 \\ 0 & 0 & 0 & \boxed{1} & 4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

First, the non-pivot columns have indices $F = \{3, 5, 6\}$, so we will construct the $n - r = 6 - 3 = 3$ vectors with a pattern of zeros and ones dictated by the indices in F. This is the realization of the first two lines of the three-case definition of the vectors $\mathbf{z}_j$, $1 \leq j \leq n-r$.

$$\mathbf{z}_1 = \begin{bmatrix} \\ \\ 1 \\ \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{z}_2 = \begin{bmatrix} \\ \\ 0 \\ \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{z}_3 = \begin{bmatrix} \\ \\ 0 \\ \\ 0 \\ 1 \end{bmatrix}.$$

Each of these vectors arises due to the presence of a non-pivot column. The remaining entries of each vector are the entries of the non-pivot column, negated, and distributed into the empty slots in order (these slots have indices in the set D, so also refer to pivot columns). This is the realization of the third line of the three-case definition of the vectors $\mathbf{z}_j$, $1 \leq j \leq n-r$.

$$\mathbf{z}_1 = \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{z}_2 = \begin{bmatrix} 1 \\ -3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{z}_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

So we have

$$\mathcal{N}(A) = \mathrm{Span}\,\{\mathbf{z}_1,\, \mathbf{z}_2,\, \mathbf{z}_3\} = \mathrm{Span}\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$