MATH1010 Midterm suggested solution

1. (a) $y = e^{x^2+1}$

   $\frac{dy}{dx} = 2xe^{x^2+1}$

   (b) $y = \frac{e^{x^2+1}}{x}$

   $\frac{dy}{dx} = \frac{(2x^2-1)e^{x^2+1}}{x^2}$

2. (a) $f(x) = \frac{|x-1|}{x}$

   $\lim_{x\to 2} |x-1| = 1$

   $\lim_{x\to 2} x = 2$

   $$\lim_{x\to 2} f(x) = \frac{1}{2}$$

   (b) for $x < 0$,

$$\begin{aligned} f(x) &= \frac{1-x}{x} \\ &= \frac{1}{x} - 1 \\ \lim_{x\to -\infty} f(x) &= \lim_{x\to -\infty} \frac{1}{x} - 1 \\ &= -1 \end{aligned}$$

3. $f(x) = xe^x$

   slope of tangent of C at $x = 1$:

$$\begin{aligned} \left.\frac{df(x)}{dx}\right|_{x=1} &= e^x + xe^x\Big|_{x=1} \\ &= 2e \end{aligned}$$

$f(1) = e$, so the required equation is

$$\frac{y-e}{x-1} = 2e$$

$$y = e(2x-1)$$

4. (a) $\lim_{x \to -\infty} x + \sqrt{x^2 + 6x + 2}$

$$= \lim_{x \to -\infty} \frac{x^2 - (x^2 + 6x + 2)}{x - \sqrt{x^2 + 6x + 2}}$$

$$= \lim_{x \to -\infty} \frac{-6x - 2}{x - \sqrt{x^2 + 6x + 2}}$$

$$= \lim_{x \to -\infty} \frac{-6 - \frac{2}{x}}{1 - \frac{\sqrt{x^2+6x+2}}{x}}$$

$$= \lim_{x \to -\infty} \frac{-6 - \frac{2}{x}}{1 - \frac{|x|}{x}\sqrt{\frac{x^2+6x+2}{x^2}}}$$

$$= -3$$

(b) Since $-1 \leqslant \sin(\frac{1}{e^x - e^{-x}}) \leqslant 1$,

we have $-|x|^3 \leqslant x^3 \sin(\frac{1}{e^x - e^{-x}}) \leqslant |x|^3$ for all $x \in \mathbb{R}$

$\lim_{x \to 0} |x|^3 = 0$ and $\lim_{x \to 0} -|x|^3 = 0$.

By Sandwich Theorem, $\lim_{x \to 0} x^3 \sin(\dfrac{1}{e^x - e^{-x}}) = 0$

5. Let $h(x) = \frac{1}{\left(f(x)\right)^2}$

$$\begin{aligned}
h'(c) &= \lim_{x \to c} \frac{h(x) - h(c)}{x - c} \\
&= \lim_{x \to c} \frac{\frac{1}{\left(f(x)\right)^2} - \frac{1}{\left(f(c)\right)^2}}{x - c} \\
&= \lim_{x \to c} -\frac{\left(f(x)\right)^2 - \left(f(c)\right)^2}{\left(f(x)\right)^2\left(f(c)\right)^2(x - c)} \\
&= \lim_{x \to c} -\frac{f(x) + f(c)}{[f(x)]^2[f(c)]^2}\frac{f(x) - f(c)}{x - c} \\
&= -\frac{2f(c)}{[f(c)]^4}f'(c) \\
&= -\frac{2f'(c)}{[f(c)]^3}
\end{aligned}$$

6.

$$
\begin{aligned}
xy + \ln(x^2 + y^2 + 100) &= 1 \\
x\frac{dy}{dx} + y + \frac{1}{x^2 + y^2 + 100}(2x + 2y\frac{dy}{dx}) &= 0 \\
\left(x + \frac{2y}{x^2 + y^2 + 100}\right)\frac{dy}{dx} &= -y - \frac{2x}{x^2 + y^2 + 100} \\
\frac{dy}{dx} &= -\frac{y(x^2 + y^2 + 100) + 2x}{x(x^2 + y^2 + 100 + 2y}
\end{aligned}
$$

7. $f(x) = |x|\sin^2 x, f(x) = \begin{cases} x\sin^2 x & x > 0 \\ 0 & x = 0 \\ -x\sin^2 x & x < 0 \end{cases}$

(a) $f'(x) = \begin{cases} \sin^2 x + 2x\sin x\cos x & x > 0 \\ -\sin^2 x - 2x\sin x\cos x & x < 0 \end{cases}$

(b) $f'(0) = \lim_{x\to 0}\frac{f(x) - f(0)}{x - 0}$

$$
\begin{aligned}
\lim_{x\to 0^+}\frac{f(x) - f(0)}{x - 0} &= \lim_{x\to 0^+}\frac{x\sin^2 x}{x} \\
&= 0 \\
\lim_{x\to 0^-}\frac{f(x) - f(0)}{x - 0} &= \lim_{x\to 0^-}\frac{-x\sin^2 x}{x} \\
&= 0
\end{aligned}
$$

$f'(0) = 0$

(c) $\lim_{x\to 0^+} f'(x) = 0, \lim_{x\to 0^-} f'(x) = 0$
Hence $f'(x)$ is continuous at $x = 0$

(d)

$$\frac{f'(x)-f'(0)}{x-0}\begin{cases}\frac{\sin^2 x+2x\sin x\cos x}{x} & x>0\\ \frac{-\sin^2 x-2x\sin x\cos x}{x} & x<0\end{cases}$$

$$\begin{aligned}\lim_{x\to 0^+}\frac{f'(x)-f'(0)}{x-0} &= \lim_{x\to 0^+}\frac{\sin^2 x}{x}+2\sin x\cos x\\ &=0\\ \lim_{x\to 0^-}\frac{f'(x)-f'(0)}{x-0} &= \lim_{x\to 0^-}-\frac{\sin^2 x}{x}-2\sin x\cos x\\ &=0\end{aligned}$$

Therefore, $f'(x)$ is differentiable at $x=0$

8. (a) Let $f(x)=x\ln x$, f is differentiable and continuous on $(0,+\infty)$
   Suppose that $0<a<b$.
   By Mean Value Theorem, $f(b)-f(a)=f'(c)(b-a)$ for some $c\in(a,b)$
   Then $b\ln b-a\ln a=(\ln c+1)(b-a)$
   Since $(\ln a+1)(b-a)<(\ln c+1)(b-1)<(\ln b+1)(b-a)$, we have $(1+\ln a)(b-a)<b\ln b-a\ln a<(1+\ln b)(b-a)$

   (b) Suppose that $0<a<b$, we have $0<\frac{1}{b}<\frac{1}{a}$
   Let $f(x)=xe^{\frac{1}{x}}$, f is differentiable and continuous on $(0,+\infty)$
   By Mean Value Theorem, $f(\frac{1}{a})-f(\frac{1}{b})=f'(d)(\frac{1}{a}-\frac{1}{b})$ for some $d\in(\frac{1}{b},\frac{1}{a})$
   Then $\frac{1}{a}e^a-\frac{1}{b}e^b=(1-\frac{1}{d})e^{\frac{1}{d}}(\frac{1}{a}-\frac{1}{b})$ Putting $c=\frac{1}{d}$, we have $c\in(a,b)$ and $\frac{1}{a}e^a-\frac{1}{b}e^b=(1-c)e^c(\frac{1}{a}-\frac{1}{b})$
   Thus $be^a-ae^b=(1-c)e^c(b-a)$ for some $c\in(a,b)$

9. (a) $f'(x)=(a^x+a^{-x})\ln a$
   For $a>1$, we have $a^x+a^{-x}>0$ and $\ln a>0$, hence $f'(x)>0$ and $f(x)$ is strictly increasing for $x>0$

   (b) Since $p-q<r-s$ and $f(x)$ is strictly increasing for $x>0$
   $a^{\frac{1}{2}(p-q)}-a^{-\frac{1}{2}(p-q)}<a^{\frac{1}{2}(r-s)}-a^{-\frac{1}{2}(r-s)}$

$$\frac{a^{\frac{1}{2}p}}{a^{\frac{1}{2}q}} - \frac{a^{\frac{1}{2}q}}{a^{\frac{1}{2}p}} = \frac{a^p - a^q}{a^{\frac{1}{2}(p+q)}}$$

$$\frac{a^p - a^q}{a^{\frac{1}{2}(p+q)}} < \frac{a^r - a^s}{a^{\frac{1}{2}(r+s)}}$$

$a^p - a^q < a^r - a^s$ as $p + q = r + s$

Putting $a = \frac{u}{v} > 1$, we have $\left(\frac{u}{v}\right)^p - \left(\frac{u}{v}\right)^q < \left(\frac{u}{v}\right)^r - \left(\frac{u}{v}\right)^s$

and then $\dfrac{u^p v^q - u^q v^p}{v^{p+q}} < \dfrac{u^r v^s - u^s v^r}{v^{r+s}}$

Therefore, $u^p v^q - u^q v^p < u^r v^s - u^s v^r$