THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
Exercise for MATH1010 University Mathematics
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Contents

1 Preliminaries

1.1 Trigonometric identities

Exercise 1. (Level 1)
Let a, b, θ be three real numbers. Prove that

$$|a\sin\theta + b\cos\theta| \leq \sqrt{a^2+b^2}.$$

Solution.

$$\begin{aligned} |a\sin\theta + b\cos\theta| &\leq \sqrt{|a\sin\theta + b\cos\theta|^2 + |a\cos\theta - b\sin\theta|^2} \\ &= \sqrt{a^2+b^2} \end{aligned}$$

□

Exercise 2. (Level 1/Level 2)
Using the well-known formula $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$, prove that

$$\cos\frac{5\pi}{12} + \cos\frac{\pi}{4} = \cos\frac{\pi}{12}.$$

Solution. Using the well-known formula $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$, we obtain that

$$\cos\frac{5\pi}{12} + \cos\frac{\pi}{4} = 2\cos\frac{\frac{5\pi}{12}+\frac{\pi}{4}}{2}\cos\frac{\frac{5\pi}{12}-\frac{\pi}{4}}{2} = 2\cos\frac{\pi}{3}\cos\frac{\pi}{12} = \cos\frac{\pi}{12}.$$

□

Exercise 3. (Level 3)
Using $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$, show that

$$\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi.$$

Solution. Using $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$, one gets

$$\begin{aligned} & \tan(\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3) \\ = & \frac{\tan(\tan^{-1} 1 + \tan^{-1} 2) + \tan(\tan^{-1} 3)}{1 - \tan^{-1} 1 + \tan^{-1} 2)\tan(\tan^{-1} 3)} \\ = & \frac{\tan(\tan^{-1} 1) + \tan(\tan^{-1} 2) + \tan(\tan^{-1} 3) - \tan(\tan^{-1} 1)\tan(\tan^{-1} 2)\tan(\tan^{-1} 3)}{1 - \tan(\tan^{-1} 1)\tan(\tan^{-1} 3) - \tan(\tan^{-1} 2)\tan(\tan^{-1} 3) - \tan(\tan^{-1} 1)\tan(\tan^{-1} 2)} \\ = & \frac{1 + 2 + 3 - 1\cdot 2\cdot 3}{1 - \tan(\tan^{-1} 1)\tan(\tan^{-1} 3) - \tan(\tan^{-1} 2)\tan(\tan^{-1} 3) - \tan(\tan^{-1} 1)\tan(\tan^{-1} 2)} \\ = & 0. \end{aligned}$$

The result follows.

□

Exercise 4. (Level 3/Level 4)
Using $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$, show that

$$\tan\frac{5\pi}{18} + \tan\frac{\pi}{3} + \tan\frac{7\pi}{18} = \tan\frac{4\pi}{9}.$$

Solution. Write $\alpha = \frac{5\pi}{18}, \beta = \frac{\pi}{3}, \gamma = \frac{7\pi}{18}$ and $\delta = \frac{\pi}{18}$. Using $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$ and $\tan\frac{\pi}{3} = \sqrt{3}$, we have

$$\begin{aligned} & \tan\alpha\tan\gamma \\ = \; & \tan(\beta - \delta)\tan(\beta + \delta) \\ = \; & \frac{\sqrt{3} - \tan\delta}{1 + \sqrt{3}\tan^2\delta}\frac{\sqrt{3} + \tan\delta}{1 - \sqrt{3}\tan^2\delta} \\ = \; & \frac{3 - \tan^2\delta}{1 - 3\tan^2\delta}. \end{aligned}$$

Iterate $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$ to get $\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan A \tan C}$. So

$$\tan(3\delta) = \frac{3\tan\delta - \tan^3\delta}{1 - 3\tan^2\delta} = \tan\delta\frac{3 - \tan^2\delta}{1 - 3\tan^2\delta} = \tan\alpha\tan\gamma\tan\delta$$

which implies the result by $\tan A = 1/\tan(\frac{\pi}{2} - A)$.

□

1.2 Mathematical Induction

Exercise 5. (Level 2)
A sequence $\{a_n\}$ is defined as follows:

$$a_1 = \frac{1}{5} \quad \text{and} \quad \frac{1}{a_{n+1}} - \frac{1}{a_n} = 2n + 5 \quad \text{for } n = 1, 2, 3, \cdots.$$

Show that $a_n = \frac{1}{n^2 + 4n}$ for $n = 1, 2, 3, \cdots$.

Solution. Let $P(n)$ be the statement that $a_n = \frac{1}{n^2 + 4n}$.

- When $n = 1$, $a_1 = \frac{1}{5} = \frac{1}{(1)^2 + 4(1)}$, hence $P(1)$ is true.

- Suppose $P(n)$ is true for some natural number n, i.e. $a_n = \frac{1}{n^2 + 4n}$ for some natural number n.
  Then,

$$\begin{aligned} \frac{1}{a_{n+1}} &= \frac{1}{a_n} + 2n + 5 \\ . &= n^2 + 4n + 2n + 5 \\ &= (n+1)^2 + 4(n+1). \end{aligned}$$

  Therefore, $a_{n+1} = \frac{1}{(n+1)^2 + 4(n+1)}$, hence $P(n+1)$ is true.

By mathematical induction, $P(n)$ is true for all natural numbers n.

□

Exercise 6. (Level 2/Level 3)

(a) Prove that, for $n \in \mathbf{N}$,

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}).$$

(b) Factorize $a^4 + a^2 + 1$.

Solution. (a) Let $P(n)$ be the statement that $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$.

- When $n = 1$, $P(1)$ is true by direct checking.
- Suppose $P(n)$ is true for some natural number n, i.e. $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$ for some natural number n.
  Then,

$$\begin{aligned} x^{n+1} - y^{n+1} &= x^{n+1} - xy^n + xy^n - y^{n+1} \\ . &= x(x^n - y^n) + y^n(x-y) \\ &= x(x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}) + y^n(x-y) \\ &= (x-y)(x^n + x^{n-1}y + \cdots + xy^{n-1} + y^n). \end{aligned}$$

  Therefore, $P(n+1)$ is true.

By mathematical induction, $P(n)$ is true for all natural numbers n.

(b) Take $x = a^2, y = 1$ and $n = 3$ in the previous part. Then we get $a^4 + a^2 + 1 = \frac{a^6-1}{a^2-1} = \frac{(a^3-1)(a^3+1)}{a^2-1} = \frac{a^3-1}{a-1}\frac{a^3+1}{a-1} = (a^2 - a + 1)(a^2 + a + 1)$.

□

Exercise 7. (Level 4)

Show that for every positive integer n, there are $r, c_r, \cdots, c_0$ such that $r\mathbf{N} \cup \{0\}$, $c_r = 1$, $c_j \in \{0, 1\}$ for $0 \leq j < r$ and

$$n = c_r 2^r + c_{r-1}2^{r-1} + \cdots + c_1 2 + c_0.$$

Remark: Such a representation is unique.

Solution. Let $P(n)$ be the statement that there are $r, c_r, \cdots, c_0$ such that $r \in \mathbf{N} \cup \{0\}$, $c_r = 1$, $c_j \in \{0, 1\}$ for $0 \leq j < r$ and

$$n = c_r 2^r + c_{r-1}2^{r-1} + \cdots + c_1 2 + c_0.$$

- When $n = 1$, take $r = 0$ and $c_0 = 1$. Then $P(1)$ is true as $1 = 02^0 + 1$.

- Suppose $P(m)$ is true for $1 \leq m < k$, i.e. there are r, $c_r, \cdots, c_0$ such that $r \in \mathbf{N} \cup \{0\}$, $c_r = 1$, $c_j \in \{0,1\}$ for $0 \leq j < r$ and

$$m = c_r 2^r + c_{r-1} 2^{r-1} + \cdots + c_1 2 + c_0.$$

We want to show that there are $r', c'_{r'}, \cdots, c'_0$ such that $r' \in \mathbf{N} \cup \{0\}$, $c'_{r'} = 1$, $c'_j \in \{0,1\}$ for $0 \leq j < r'$ and

$$k = c'_{r'} 2^r + c'_{r'-1} 2^{r-1} + \cdots + c'_1 2 + c'_0.$$

Consider two cases according to the parity of k. First, suppose k is even. By induction hypothesis, there are r, $c_r, \cdots, c_0$ such that $r \in \mathbf{N} \cup \{0\}$, $c_r = 1$, $c_j \in \{0,1\}$ for $0 \leq j < r$ and

$$\frac{k}{2} = c_r 2^r + c_{r-1} 2^{r-1} + \cdots + c_1 2 + c_0,$$

then

$$k = c_r 2^{r+1} + c_{r-1} 2^r + \cdots + c_1 2^2 + c_0 2.$$

We set $r' = r + 1$, $c'_0 = 0$ and $c'_{j+1} = c_j$ for $0 \leq j \leq r$. Thus, there are $r', c'_{r'}, \cdots, c'_0$ such that $r' \in \mathbf{N} \cup \{0\}$, $c'_{r'} = 1$, $c'_j \in \{0,1\}$ for $0 \leq j < r'$ and

$$k = c'_{r'} 2^r + c'_{r'-1} 2^{r-1} + \cdots + c'_1 2 + c'_0.$$

Now suppose that k is odd, that is, $k = 2m + 1$ for some $m < k$. Apply the induction hypothesis on m. There are r, $c_r, \cdots, c_0$ such that $r \in \mathbf{N} \cup \{0\}$, $c_r = 1$, $c_j \in \{0,1\}$ for $0 \leq j < r$ and

$$m = c_r 2^r + c_{r-1} 2^{r-1} + \cdots + c_1 2 + c_0,$$

then

$$k = c_r 2^{r+1} + c_{r-1} 2^r + \cdots + c_1 2^2 + c_0 2 + 1.$$

We set $r' = r + 1$, $c'_0 = 1$ and $c'_{j+1} = c_j$ for $0 \leq j \leq r$. Thus, there are $r', c'_{r'}, \cdots, c'_0$ such that $r' \in \mathbf{N} \cup \{0\}$, $c'_{r'} = 1$, $c'_j \in \{0,1\}$ for $0 \leq j < r'$ and

$$k = c'_{r'} 2^r + c'_{r'-1} 2^{r-1} + \cdots + c'_1 2 + c'_0.$$

Therefore, $P(k)$ is true.

By mathematical induction, $P(n)$ is true for all natural numbers n.

$\square$

2 Sequences

2.1 Intuitive definition of limits of sequences

Exercise 1. (Level 2)
Evaluate the following limits.

(a) $\lim\limits_{n\to\infty} \frac{n^5+n^2-n+1}{3n^5+4(n+1)^4+2(2n-1)^2}$

(b) $\lim\limits_{n\to\infty} (\sqrt[3]{n^2+1} - \sqrt[3]{n^2})$

(c) $\lim\limits_{n\to\infty} \{\left[(n+1)^{\frac{3}{2}} - n^{\frac{3}{2}}\right](\sqrt{n+1} - \sqrt{n})\}$

(d) $\lim\limits_{n\to\infty} \left[\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right)\right]$

Solution. (a)

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^5+n^2-n+1}{3n^5+4(n+1)^4+2(2n-1)^2} &= \lim_{n\to\infty} \frac{1+\frac{1}{n^3}-\frac{1}{n^4}+\frac{1}{n^5}}{3+\frac{4}{n}(1+\frac{1}{n})^4+\frac{2}{n^3}(2-\frac{1}{n})^2} \\
&= \frac{1}{3}
\end{aligned}$$

(b)

$$\begin{aligned}
& \lim_{n\to\infty} (\sqrt[3]{n^2+1} - \sqrt[3]{n^2}) \\
= & \lim_{n\to\infty} (\sqrt[3]{n^2+1} - \sqrt[3]{n^2}\frac{\sqrt[3]{(n^2+1)^2}+\sqrt[3]{n^2(n^2+1)}+\sqrt[3]{n^4}}{\sqrt[3]{(n^2+1)^2}+\sqrt[3]{n^2(n^2+1)}+\sqrt[3]{n^4}}) \\
= & \lim_{n\to\infty} \frac{1}{\sqrt[3]{(n^2+1)^2}+\sqrt[3]{n^2(n^2+1)}+\sqrt[3]{n^4}} \\
= & \ 0
\end{aligned}$$

(c)

$$\begin{aligned}
& \lim_{n\to\infty} \{\left[(n+1)^{\frac{3}{2}} - n^{\frac{3}{2}}\right](\sqrt{n+1} - \sqrt{n})\} \\
= & \lim_{n\to\infty} \{\left[(n+1)^{\frac{3}{2}} - n^{\frac{3}{2}}\right]\frac{(n+1)^{\frac{3}{2}} + n^{\frac{3}{2}}}{(n+1)^{\frac{3}{2}} + n^{\frac{3}{2}}}(\sqrt{n+1} - \sqrt{n})\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\} \\
= & \lim_{n\to\infty} \frac{3n^2+3n+1}{((n+1)^{\frac{3}{2}} + n^{\frac{3}{2}})(\sqrt{n+1}+\sqrt{n})} \\
= & \lim_{n\to\infty} \frac{3+\frac{3}{n}+\frac{1}{n^2}}{((1+\frac{1}{n})^{\frac{3}{2}}+1)(\sqrt{1+\frac{1}{n}}+1)} \\
= & \ \frac{3}{4}
\end{aligned}$$

(d)

$$\begin{aligned}
& \lim_{n\to\infty}\left[\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right)\right] \\
= & \lim_{n\to\infty}\left[\left(1-\frac{1}{2}\right)\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{3}\right)\cdots\left(1-\frac{1}{n}\right)\left(1+\frac{1}{n}\right)\right] \\
= & \lim_{n\to\infty}\left[\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{2}{3}\right)\left(\frac{4}{3}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{n+1}{n}\right)\right] \\
= & \lim_{n\to\infty}\left[\left(\frac{1}{2}\right)\left(\frac{n+1}{n}\right)\right] \\
= & \frac{1}{2}
\end{aligned}$$

□

Exercise 2. (Level 2)
Let $a \in \mathbf{R}$. A sequence $\{u_n\}$ is defined by

$$u_n = \frac{a^n}{1+a^{n+1}}.$$

(a) Show that $u_n = \frac{1}{a}\left[1 - \frac{1}{1+a^{n+1}}\right]$.

(b) Discuss the behaviour, as $n \to \infty$, of the sequence $\{u_n\}$ for the cases

(i) $|a| > 1$,

(ii) $|a| < 1$,

(iii) $a = 1$,

(iv) $a = -1$.

(Hint: You may use $\lim_{n\to\infty} a^n = \pm\infty$ for $|a| > 1$ and $\lim_{n\to\infty} a^n = 0$ for $|a| < 1$.)

Solution. (a) Straightforward.

(b) (i) For $|a| > 1$, using the well-known limits $\lim_{n\to\infty} a^n = \pm\infty$, one gets

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{a}\left[1 - \frac{1}{1+a^{n+1}}\right] = \frac{1}{a}.$$

(ii) For $|a| < 1$, using the well-known limit $\lim_{n\to\infty} a^n = 0$, we have

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{a}\left[1 - \frac{1}{1+a^{n+1}}\right] = 0.$$

(iii) For $a = 1$, $u_n = \frac{1}{2}$ for all positive integers n, thus, $\lim_{n\to\infty} u_n = \frac{1}{2}$.

(iv) For $a = -1$, the sequence becomes

$$u_n = \begin{cases} -\frac{1}{2} & \text{if } n \text{ is odd} \\ \text{undefined} & \text{if } n \text{ is even.} \end{cases}$$

The limit does not exist.

□

2.2 Monotone Convergence theorem

Exercise 3. (Level 2)
For each of the following sequence,

(i) Determine whether the sequence is monotonic;

(ii) Determine whether the sequence is bounded.

(a) $a_n = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n}$

(b) $a_n = \frac{\sqrt{n^2+1}-n}{n^2}$

Solution. (a) (i) For all $n \geq 1$,

$$\begin{aligned} a_{n+1} - a_n &= \frac{1}{n+1} + \frac{1}{n+1} + \cdots + \frac{1}{2n+2} - \left(\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n}\right) \\ &= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n} \\ &< \frac{1}{2n} + \frac{1}{2n} - \frac{1}{n} \\ &= 0 \end{aligned}$$

implying that $\{a_n\}$ is monotonic (strictly) decreasing.

(ii) Noting that

$$0 < \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n} < \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n} < \frac{n+1}{n},$$

this shows that $\{a_n\}$ is bounded.

(b) $a_n = \frac{\sqrt{n^2+1}-n}{n^2}$

(i) For all $n \geq 1$,

$$\begin{aligned} &\frac{a_{n+1}}{a_n} \\ &= \frac{\sqrt{(n+1)^2+1} - (n+1)}{(n+1)^2} \frac{n^2}{\sqrt{n^2+1} - n} \\ &= \frac{n^2}{(n+1)^2} \frac{\sqrt{(n+1)^2+1}-(n+1)}{\sqrt{n^2+1}-n} \frac{\sqrt{(n+1)^2+1}+(n+1)}{\sqrt{n^2+1}+n} \frac{\sqrt{n^2+1}+n}{\sqrt{(n+1)^2+1}+(n+1)} \\ &= \frac{n^2}{(n+1)^2} \frac{\sqrt{n^2+1}+n}{\sqrt{(n+1)^2+1}+(n+1)} \\ &< 1 \end{aligned}$$

implying that $\{a_n\}$ is monotonic (strictly) decreasing.

(ii) Noting that for $n \geq 1$

$$0 = \frac{\sqrt{n^2} - n}{n^2} < \frac{\sqrt{n^2+1} - n}{n^2} < \frac{\sqrt{n^2+n^2}}{n^2} < \frac{\sqrt{2}}{n} < 3,$$

this shows that $\{a_n\}$ is bounded.

□

Exercise 4. (Level 3)
A sequence is defined by $x_1 = 1, x_{n+1} = \frac{2}{3}x_n + \frac{9}{x_n^2}$ for $n \geq 1$.

(a) (i) Show that $x_n > 3$ for $n \geq 2$.
(ii) Prove that $x_{n+1} < x_n$ for $n \geq 2$.

(b) Hence show that $\{x_n\}$ converges and find $\lim_{n\to\infty} x_n$.

Solution. (a) (i) Consider $x_{n+1} - 3$. Then,

$$\begin{aligned} x_{n+1} - 3 &= \frac{2}{3}x_n + \frac{9}{x_n^2} - 3 \\ &= \frac{2x_n^3 - 9x_n^2 + 27}{3x_n^2} \\ &= \frac{(x_n - 3)^2(2x_n + 3)}{3x_n^2}. \end{aligned}$$

Note that $x_2 > 3$. Thus $x_{n+1} - 3 > 0$ for $n \geq 2$. Hence $x_n > 3$ for $n \geq 2$.

(ii) Noting that

$$\begin{aligned} x_{n+1} - x_n &= \frac{2}{3}x_n + \frac{9}{x_n^2} - x_n \\ &= \frac{27 - x_n^3}{3x_n^2} \\ &= \frac{(3 - x_n)(x_n^2 + x_n + 9)}{3x_n^2}, \end{aligned}$$

one gets $x_{n+1} < x_n$ for $n \geq 2$ provided that $x_n > 3$ for $n \geq 2$.

(b) From parts (i) and (ii) of (a), $\{x_n\}$ converges by monotone convergence theorem. Let $\lim_{n\to\infty} x_n = L$. Then, by the convergence of sub-sequence,

$$\begin{aligned} \lim_{n\to\infty} x_{n+1} &= \lim_{n\to\infty} \left(\frac{2}{3}x_n + \frac{9}{x_n^2}\right) \\ L &= \frac{2}{3}L + \frac{9}{L^2} \end{aligned}$$

gives $L^3 = 27$. On solving, it gives a real solution $L = 3$. Hence $\lim_{n\to\infty} x_n = 3$.

□

Exercise 5. (Level 4)
Let a_1 and b_1 be any two positive numbers such that $a_1 > b_1$. Let $\{a_n\}$ and $\{b_n\}$ be two sequences satisfying the equations

$$a_{n+1} = \frac{a_n + b_n}{2} \quad \text{and} \quad b_{n+1} = \sqrt{a_n b_n}$$

for all positive integers n.

(a) (i) Show that $a_n > b_n$ for for all positive integers n.

(ii) Deduce that $\{a_n\}$ is monotonic decreasing and $\{b_n\}$ is monotonic increasing.

(b) Show that both $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist. Hence prove that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

Solution. (a) (i) We note that

$$a_{n+1} - b_{n+1} = \frac{a_n + b_n}{2} - \sqrt{a_n b_n} = \frac{(\sqrt{a_n} + \sqrt{b_n})^2}{2} > 0$$

for $n \geq 1$. The results follows.

(ii) For $n \geq 1$, by previous part, we get

$$a_{n+1} = \frac{a_n + b_n}{2} < \frac{a_n + a_n}{2} = a_n,$$

that is, $\{a_n\}$ is monotonic decreasing. Also, by similar arguments, one obtains, for $n \geq 1$,

$$b_{n+1} = \sqrt{a_n b_n} > \sqrt{b_n \cdot b_n} = b_n$$

which says that $\{b_n\}$ is monotonic increasing.

(b) From (a), we note that for $n \geq 1$

$$a_1 > a_{n+1} > b_{n+1} > b_1.$$

We see that $\{a_n\}$ is monotonic decreasing and bounded below while $\{a_n\}$ is monotonic increasing and bounded above. By monotone convergence theorem, both $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist. Let $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then, by the recurrence relation, we have

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{a_n + b_n}{2}$$

so $A = \frac{A+B}{2}$, hence $A = B$. Therefore, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ is obtained.

□

Exercise 6. (Level 4)

The two sequences of positive integers $\{a_1, a_2, \cdots, a_n, \cdots\}$, $\{b_1, b_2, \cdots, b_n, \cdots\}$ satisfy the following conditions:

$$a_1 = b_1 = 1 \quad \text{and} \quad a_{n+1} = a_n + 2b_n,\ b_{n+1} = a_n + b_n$$

for all positive integers n.

(a) Prove that for each positive integer n,

$$a_n \geq n,\ b_n \geq n \quad \text{and} \quad a_n^2 - b_n^2 = (-1)^n.$$

(b) (i) Deduce from (a) that

$$\frac{a_n}{b_n} < \sqrt{2} \quad \text{if } n \text{ is odd}, \quad \frac{a_n}{b_n} > \sqrt{2} \quad \text{if } n \text{ is even},$$

and

$$\frac{a_{n+2}}{b_{n+2}} - \frac{a_n}{b_n} = \frac{2(-1)^{n+1}}{b_n(2a_n + 3b_n)}.$$

(ii) Hence show that the two sequences $\{\frac{a_{2k-1}}{b_{2k-1}} : k = 1, 2, 3 \cdots\}$ and $\{\frac{a_{2k}}{b_{2k}} : k = 1, 2, 3 \cdots\}$ converge to the same limit $\sqrt{2}$.

Solution. (a) For $n = 1$, $a_1 = b_1 = 1$.

Suppose that $a_n \geq n$, $b_n \geq n$. Then

$$a_{n+1} = a_n + 2b_n \geq n + 2n \geq n + 1 \quad \text{and} \quad b_{n+1} = a_n + b_n \geq n + n \geq n + 1.$$

By the principle of induction, the results are true for all positive integers n. Next, noting that $a_{n+1}^2 - 2b_{n+1}^2 = (a_n + 2b_n)^2 - 2(a_n + b_n)^2 = -(a_n^2 - 2b_n^2)$, one gives that

$$\begin{aligned} a_n^2 - 2b_n^2 &= -(a_{n-1}^2 - 2b_{n-1}^2) \\ &\vdots \\ &= (-1)^{n-1}(a_1^2 - 2b_1^2) \\ &= (-1)^n. \end{aligned}$$

(b) (i) Using (a), if n is odd, one gets $a_n^2 - 2b_n^2 = -1 < 0$ implying that $\frac{a_n}{b_n} < \sqrt{2}$ and if n is even, one gets $a_n^2 - 2b_n^2 = 1 > 0$ implying that (by (a), $\frac{a_n}{b_n} > 0$) $\frac{a_n}{b_n} > \sqrt{2}$.

Again, by (a), we see that

$$\begin{aligned} \frac{a_{n+2}}{b_{n+2}} - \frac{a_n}{b_n} &= \frac{a_{n+1} + 2b_{n+1}}{a_{n+1} + b_{n+1}} - \frac{a_n}{b_n} \\ &= \frac{a_n + b_n + 2a_n + 4b_n}{a_n + b_n + a_n + 2b_n} - \frac{a_n}{b_n} \\ &= -\frac{2(a_n^2 - 2b_n^2)}{b_n(2a_n + 3b_n)} \\ &= \frac{2(-1)^{n+1}}{b_n(2a_n + 3b_n)}. \end{aligned}$$

(ii) From above results, we get (by (a), $\frac{a_n}{b_n} > 0$) $\frac{a_{2k+1}}{b_{2k+1}} - \frac{a_{2k-1}}{b_{2k-1}} = \frac{2}{b_{2k-1}(2a_{2k-1}+3b_{2k-1})} > 0$ and $\frac{a_{2k}}{b_{2k}} - \frac{a_{2k-2}}{b_{2k-2}} = \frac{-2}{b_{2k-2}(2a_{2k-2}+3b_{2k-2})} < 0$. Thus, $\{\frac{a_{2k-1}}{b_{2k-1}} : k = 1, 2, 3 \cdots\}$ is monotonic increasing and $\{\frac{a_{2k}}{b_{2k}} : k = 1, 2, 3 \cdots\}$ is monotonic decreasing. By (b)(i), $\{\frac{a_{2k-1}}{b_{2k-1}} : k = 1, 2, 3 \cdots\}$ is bounded above by $\sqrt{2}$ and $\{\frac{a_{2k}}{b_{2k}} : k = 1, 2, 3 \cdots\}$ is bounded below by $\sqrt{2}$, therefore, by monotone convergence theorem, both $\{\frac{a_{2k-1}}{b_{2k-1}}\}$ and $\{\frac{a_{2k}}{b_{2k}}\}$ converge. Let $\lim_{k\to\infty} \frac{a_{2k-2}}{b_{2k-2}} = L_1$ and $\lim_{k\to\infty} \frac{a_{2k}}{b_{2k}} = L_2$ then

$$\lim_{k\to\infty} \frac{a_{2k-1}}{b_{2k-1}} = \lim_{k\to\infty} \frac{a_{2k-2} + 2b_{2k-2}}{a_{2k-2} + b_{2k-2}} = \lim_{k\to\infty} \frac{\frac{a_{2k-2}}{b_{2k-2}} + 2}{\frac{a_{2k-2}}{b_{2k-2}} + 1}$$

and

$$\lim_{k\to\infty} \frac{a_{2k}}{b_{2k}} = \lim_{k\to\infty} \frac{a_{2k-1} + 2b_{2k-1}}{a_{2k-1} + b_{2k-1}} = \lim_{k\to\infty} \frac{\frac{a_{2k-1}}{b_{2k-1}} + 2}{\frac{a_{2k-1}}{b_{2k-1}} + 1}.$$

These set up two equations

$$L_1 = \frac{L_2 + 2}{L_2 + 1} \quad \text{and} \quad L_2 = \frac{L_1 + 2}{L_1 + 1}.$$

On solving, one obtains (by (a), $\frac{a_n}{b_n} > 0$ implies that L_1 and L_2 both are non-negative) $L_1 = L_2 = \sqrt{2}$, hence the results follow.

□

2.3 Sandwich theorem for sequences

Exercise 7. (Level 3)
Evaluate the following limits

(a) $\lim\limits_{n\to\infty} \frac{\sqrt{n+1}}{n}$

(b) $\lim\limits_{n\to\infty} \frac{(-1)^n n^2 - 1}{n^3}$

(c) $\lim\limits_{n\to\infty} \frac{\sin^2(n!)}{n!}$

(d) $\lim\limits_{n\to\infty} \frac{\sinh^2(n^{-2})}{n}$

Solution. (a) Note that for any positive integer n,

$$0 < \frac{\sqrt{n+1}}{n^{2n}} < \frac{\sqrt{2n}}{n^{2n}} < \frac{\sqrt{2}}{n}.$$

Together with $\lim\limits_{n\to\infty} \frac{1}{n} = 0$, by sandwich theorem, we get $\lim\limits_{n\to\infty} \frac{\sqrt{n+1}}{n^{2n}} = 0$.

(b) Observe that for any positive integer n,

$$\frac{-n^2 - 1}{n^3} < \frac{(-1)^n n^2 - 1}{n^3} < \frac{n^2 - 1}{n^3}.$$

With this fact $\lim\limits_{n\to\infty} \frac{-n^2-1}{n^3} = \lim\limits_{n\to\infty} \frac{n^2-1}{n^3} = 0$, by sandwich theorem, one has $\lim\limits_{n\to\infty} \frac{(-1)^n n^2 - 1}{n^3} = 0$.

(c) Since $0 < \sin^2(n!) < 1$, then we have that $0 < \frac{\sin^2(n!)}{n!} < \frac{1}{n!} \leq \frac{1}{n}$ for all positive integer n. Hence $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ gives $\lim\limits_{n\to\infty} \frac{\sin^2(n!)}{n!} = 0$ by the sandwich theorem.

(d) Recall that $\sinh x = \frac{e^x - e^{-x}}{2}$, then $0 < \sinh^2(n^{-2}) < \frac{e-1}{2} < 1$ for all positive integer n. Thus, it implies that

$$0 < \frac{\sinh^2(n^{-2})}{n} < \frac{1}{n}.$$

Therefore, because $\lim\limits_{n\to\infty} \frac{1}{n} = 0$, $\lim\limits_{n\to\infty} \frac{\sinh^2(n^{-2})}{n} = 0$ by sandwich theorem.

□

Exercise 8. (Level 3)

(a) For $1 \leq k \leq n$, show that $(n+1-k)k \geq n$. Hence, deduce that $(n!)^2 \geq n^n$.

(b) Find $\lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}}$, if exists.

Solution. (a) Observe that, for $1 \leq k \leq n$, $(n+1-k)k - n = (n-k)(k-1) \geq 0$, that is $(n+1-k)k \geq n$. Multiplying the inequalities $(n+1-k)k \geq n$ from $k=1$ to $k=n$, we then get

$$(n!)^2 \geq n^n.$$

(b) From the previous part, one has $0 < \frac{1}{\sqrt[n]{n!}} \leq \frac{1}{\sqrt{n}}$. Since $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, by the sandwich theorem, $\lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}} = 0$.

□

Exercise 9. (Level 3)

Let $s_n = \frac{1}{(2n)^2} + \frac{1}{(2n+1)^2} + \cdots + \frac{1}{(3n)^2}$, show that $\lim_{n\to\infty} s_n = 0$.

Solution. Observe that

$$\frac{2n+1}{(3n)^2} \leq \frac{1}{(3n)^2} + \frac{1}{(3n)^2} + \cdots + \frac{1}{(3n)^2} \leq \frac{1}{(2n)^2} + \frac{1}{(2n+1)^2} + \cdots + \frac{1}{(3n)^2}$$

and

$$\frac{1}{(2n)^2} + \frac{1}{(2n+1)^2} + \cdots + \frac{1}{(3n)^2} \leq \frac{1}{(2n)^2} + \frac{1}{(2n)^2} + \cdots + \frac{1}{(2n)^2} \leq \frac{2n+1}{(3n)^2}.$$

Then $\frac{2n+1}{(3n)^2} \leq s_n \leq \frac{2n+1}{(2n)^2}$, together with $\lim_{n\to\infty} \frac{2n+1}{(2n)^2} = \lim_{n\to\infty} \frac{2n+1}{(3n)^2} = 0$, by the sandwich theorem, $\lim_{n\to\infty} s_n = 0$ as desired. □

Exercise 10. (Level 4/Level 5)

(a) Let x be a positive real number. Prove that for any positive integer n greater than 1,

$$(1+x)^n > 1 + nx.$$

(b) Let t be any fixed positive number. Consider the sequence

$$a_1, a_2, \cdots, a_n, \cdots$$

where $a_n = \sqrt[n]{t}$.

(i) Let $t > 1$.

A. Show that $a_n > 1$.

B. Putting $a_n = 1 + x_n$, show that, for $n \geq 2$,

$$1 < a_n < 1 + \frac{t-1}{n}.$$

C. Show that for $t > 1$, $\lim_{t\to\infty} a_n = 1$.

(ii) Show that for $t > 0$, $\lim_{t\to\infty} a_n = 1$.

(c) For any k fixed positive numbers $a_1, a_2, \cdots, a_k$, prove that the sequence

$$v_n = \sqrt[n]{a_1^n + a_2^n + \cdots + a_k^n}$$

converges and find its limit.

Solution. (a) For any positive integer n greater than 1, using binomial theorem, one gets

$$(1+x)^n = 1 + nx + C_2^n x^2 + \cdots + x^n > 1 + nx.$$

(b) (i) A. Since $t > 1$, $a_n > \sqrt[n]{t} > 1$.

B. By the previous result, $a_n > 1$ for $n \geq 2$. For the other inequality, consider $(1+x_n)^n$. Using part (a), we have $t = (1+x_n)^n > 1 + nx_n$, therefore, $x_n < \frac{t-1}{n}$ for $n \geq 2$. Hence $1 < a_n = 1 + x_n < 1 + \frac{t-1}{n}$ for $n \geq 2$.

C. Since $\lim_{n\to+\infty} \frac{t-1}{n} = 0$, by sandwich theorem, we get $\lim_{t\to\infty} a_n = 1$.

(ii) We consider three cases:

- For $t > 1$, by (b)(i), we get $\lim_{t\to\infty} a_n = 1$.
- For $t = 1$, $a_n = 1$ for all positive integers n. Then, $\lim_{t\to\infty} a_n = 1$.
- For $0 < t < 1$, Consider $b_n := \frac{1}{a_n} = \sqrt[n]{\frac{1}{t}}$ for all positive integers n. Since $\frac{1}{t} > 1$, $\lim_{t\to\infty} b_n = 1$, by (b)(i), and hence $\lim_{t\to\infty} a_n = 1$.

(c) Let a be the largest of the numbers $a_1, a_2, \cdots, a_k$, then

$$a < \sqrt[n]{a_1^n + a_2^n + \cdots + a_k^n} \leq a\sqrt[n]{k}.$$

By above result, $\lim_{t\to\infty} \sqrt[n]{k} = 1$, therefore, by sandwich theorem, we get

$$\lim_{t\to\infty} \sqrt[n]{a_1^n + a_2^n + \cdots + a_k^n} = a.$$

□

3 Limits of functions

3.1 Intuitive definition of limits of functions

Exercise 1. (Level 2)
Evaluate the following limits.

(a) $\lim_{x\to-1} \frac{x^3-2x-1}{x^5-2x-1}$

(b) $\lim_{x\to1} \frac{x+x^2+x^3+\cdots+x^n-n}{x-1}$

(c) $\lim_{x\to+\infty} \frac{x^2+3}{x\sqrt{x^2+1}}$

(d) $\lim_{x\to-\infty} \frac{x^2+3}{x\sqrt{x^2+1}}$

(e) $\lim_{x\to+\infty} \left[x^{\frac{3}{2}}\left(\sqrt{x+2}-2\sqrt{x+1}+\sqrt{x}\right)\right]$

Solution. (a)

$$\begin{aligned}\lim_{x\to-1}\frac{x^3-2x-1}{x^5-2x-1} &= \lim_{x\to-1}\frac{(x+1)(x^2-x-1)}{(x+1)(x^4-x^3+x^2-x-1)}\\ &= \lim_{x\to-1}\frac{(x^2-x-1)}{(x^4-x^3+x^2-x-1)}\\ &= \frac{1}{3}\end{aligned}$$

(b) Note that for any positive integer k, $\lim_{x\to1}\frac{x^k-1}{x-1} = \lim_{x\to1}\frac{(x-1)(x^{k-1}+x^{k-2}+\cdots+1)}{x-1} = k$. Then,

$$\begin{aligned}&\lim_{x\to1}\frac{x+x^2+x^3+\cdots+x^n-n}{x-1}\\ =&\lim_{x\to1}\frac{(x-1)+(x^2-1)+(x^3-1)+\cdots+(x^n-1)}{x-1}\\ =&\lim_{x\to1}\frac{x-1}{x-1}+\lim_{x\to1}\frac{x^2-1}{x-1}+\lim_{x\to1}\frac{x^3-1}{x-1}+\cdots+\lim_{x\to1}\frac{x^n-1}{x-1}\\ =&\ 1+2+3+\cdots+n\\ =&\ \frac{n(n+1)}{2}.\end{aligned}$$

(c)

$$\lim_{x\to+\infty}\frac{x^2+3}{x\sqrt{x^2+1}} = \lim_{x\to+\infty}\frac{1+\frac{3}{x^2}}{\sqrt{1+\frac{1}{x^2}}} = 1$$

(d) We have, by the substitution, $y=-x$, $y\to+\infty$ as $x\to-\infty$,

$$\lim_{x\to-\infty}\frac{x^2+3}{x\sqrt{x^2+1}} = \lim_{y\to+\infty}-\frac{y^2+3}{y\sqrt{y^2+1}} = \lim_{y\to+\infty}-\frac{1+\frac{3}{y^2}}{\sqrt{1+\frac{1}{y^2}}} = -1.$$

(e)

$$\begin{aligned}
&\lim_{x\to+\infty}\left[x^{\frac{3}{2}}\left(\sqrt{x+2}-2\sqrt{x+1}+\sqrt{x}\right)\right]\\
=&\lim_{x\to+\infty}\left[x^{\frac{3}{2}}\left(\sqrt{x+2}-\sqrt{x+1}+\sqrt{x}-\sqrt{x+1}\right)\right]\\
=&\lim_{x\to+\infty}\left[x^{\frac{3}{2}}\left(\frac{(\sqrt{x+2}-\sqrt{x+1})(\sqrt{x+2}+\sqrt{x+1})}{\sqrt{x+2}+\sqrt{x+1}}+\frac{(\sqrt{x}-\sqrt{x+1})(\sqrt{x}+\sqrt{x+1})}{\sqrt{x}+\sqrt{x+1}}\right)\right]\\
=&\lim_{x\to+\infty}\left[x^{\frac{3}{2}}\left(\frac{1}{\sqrt{x+2}+\sqrt{x+1}}-\frac{1}{\sqrt{x}+\sqrt{x+1}}\right)\right]\\
=&\lim_{x\to+\infty}\left[x^{\frac{3}{2}}\left(\frac{(\sqrt{x}-\sqrt{x+2})}{(\sqrt{x+2}+\sqrt{x+1})(\sqrt{x}+\sqrt{x+1})}\right)\right]\\
=&\lim_{x\to+\infty}\left[x^{\frac{3}{2}}\left(\frac{(\sqrt{x}-\sqrt{x+2})(\sqrt{x}+\sqrt{x+2})}{(\sqrt{x+2}+\sqrt{x+1})(\sqrt{x}+\sqrt{x+1})(\sqrt{x}+\sqrt{x+2})}\right)\right]\\
=&\lim_{x\to+\infty}\left[x^{\frac{3}{2}}\left(\frac{-2}{(\sqrt{x+2}+\sqrt{x+1})(\sqrt{x}+\sqrt{x+1})(\sqrt{x}+\sqrt{x+2})}\right)\right]\\
=&\lim_{x\to+\infty}\left(\frac{-2}{(\sqrt{1+\frac{2}{x}}+\sqrt{1+\frac{1}{x}})(1+\sqrt{1+\frac{1}{x}})(1+\sqrt{1+\frac{2}{x}})}\right)\\
=&-\frac{1}{4}
\end{aligned}$$

□

3.2 Sandwich theorem for functions

Exercise 2. (Level 2)
Evaluate the following limits.

(a) $\lim\limits_{x\to0^-} x|\sin\frac{1}{x}|$

(b) $\lim\limits_{x\to+\infty}\frac{\sin\tan x+\tan\sin x}{x+1}$

(c) $\lim\limits_{x\to+\infty}\frac{x^5+x^3\sin\frac{1}{x^2}}{x^5+7}$

(d) $\lim\limits_{x\to+\infty}\frac{3x^2+\sin 5x+\cos 7x}{11x^2+\sin 13x+\cos 17x}$

Solution. (a) Since for $x<0$, $0<|\sin\frac{1}{x}|<1$, we get $x<x|\sin\frac{1}{x}|<0$ for $x<0$. Because $\lim\limits_{x\to0^-}x=0$, by the sandwich theorem, $\lim\limits_{x\to0^-}x|\sin\frac{1}{x}|=0$.

(b) Provided that x is large enough, we have $-1<\sin\tan x<1$ and $-tan1<\tan\sin x<tan1$. Then one has $-\frac{1+tan1}{x+1}<\frac{\sin\tan x+\tan\sin x}{x+1}<\frac{1+tan1}{x+1}$. Because $\lim\limits_{x\to+\infty}\frac{1+tan1}{x+1}=0$, by the sandwich theorem, $\lim\limits_{x\to+\infty}\frac{\sin\tan x+\tan\sin x}{x+1}=0$

(c) When x is very large,

$$x^5 - x^3 \leq x^5 + x^3 \sin\frac{1}{x^2} \leq x^5 + x^3.$$

Hence we get

$$\frac{x^5 - x^3}{x^5+7} \leq \frac{x^5 + x^3 \sin\frac{1}{x^2}}{x^5+7} \leq \frac{x^5+x^3}{x^5+7}.$$

Let $g(x) = \frac{x^5-x^3}{x^5+7}$ and $\frac{x^5+x^3}{x^5+7}$.

$$\lim_{x\to+\infty} g(x) = \lim_{x\to+\infty} h(x) = 1,$$

thus, by the sandwich theorem, we have $\lim_{x\to+\infty} \frac{x^5+x^3\sin\frac{1}{x^2}}{x^5+7} = 1$.

(d) By sandwich theorem, we have $\lim_{x\to+\infty} \frac{\sin nx}{x^2} = 0$ and $\lim_{x\to+\infty} \frac{\cos mx}{x^2} = 0$ for any n, m. Hence $\lim_{x\to+\infty} \dfrac{3x^2+\sin 5x+\cos 7x}{11x^2+\sin 13x+\cos 17x} = \lim_{x\to+\infty} \dfrac{3+\frac{\sin 5x}{x^2}+\frac{\cos 7x}{x^2}}{11+\frac{\sin 13x}{x^2}+\frac{\cos 17x}{x^2}} = \frac{3}{11}$.

□

3.3 Two important limits: $\lim_{x\to+\infty}(1+\frac{1}{x})^x = e$ and $\lim_{x\to 0}\frac{\sin x}{x} = 1$

Exercise 3. (Level 3)
Evaluate the following limits.

(a) $\lim_{x\to 0} \frac{1+\sin x-\cos x}{1+\sin px-\cos px}$

(b) $\lim_{x\to 0} \frac{\cot(a+2x)-2\cot(a+x)+\cot a}{x^2}$

(c) $\lim_{x\to 0} \frac{x^2}{\sqrt{1+x\sin x}-\sqrt{\cos x}}$

(d) $\lim_{x\to 0} \frac{1-\cos x\cos 2x\cos 3x}{1-\cos x}$

(e) $\lim_{x\to+\infty} \frac{(x+a)^{x+a}(x+b)^{x+b}}{(x+a+b)^{2x+a+b}}$

(f) $\lim_{x\to 0} \frac{\sqrt{1+x\sin x}-1}{e^{x^2}-1}$

Solution. (a)

$$\begin{aligned}\lim_{x\to 0}\frac{1+\sin x-\cos x}{1+\sin px-\cos px} &= \lim_{x\to 0}\frac{2\sin^2\frac{x}{2}+\sin x}{2\sin^2\frac{px}{2}+\sin px} \\ &= \lim_{x\to 0}\frac{2\frac{\sin^2\frac{x}{2}}{x}+\frac{\sin x}{x}}{2\frac{\sin^2\frac{px}{2}}{x}+\frac{\sin px}{x}} \\ &= \frac{1}{p}\end{aligned}$$

(b) Note that, by trigonometry identities,

$$\begin{aligned}
& \cot(a+2x) - 2\cot(a+x) + \cot a \\
= \; & \cot(a+2x) - \cot(a+x) + \cot a - \cot(a+x) \\
= \; & \tfrac{\cos(a+2x)\sin(a+x) - \sin(a+2x)\cos(a+x)}{\sin(a+2x)\sin(a+x)} + \tfrac{\cos(a)\sin(a+x) - \sin(a)\cos(a+x)}{\sin(a)\sin(a+x)} \\
= \; & \frac{-\sin x}{\sin(a+2x)\sin(a+x)} + \frac{\sin x}{\sin(a)\sin(a+x)} \\
= \; & \frac{\sin x}{\sin(a+x)} \cdot \frac{\sin(a+2x) - \sin(a)}{\sin(a+2x)\sin(a)} \\
= \; & \frac{\sin x}{\sin(a+x)} \cdot \frac{2\cos(a+x)\sin x}{\sin(a+2x)\sin(a)} \\
= \; & \frac{2\cos(a+x)\sin^2 x}{\sin(a+2x)\sin(a+x)\sin(a)}.
\end{aligned}$$

Hence, it implies that

$$\begin{aligned}
& \lim_{x\to 0} \frac{\cot(a+2x) - 2\cot(a+x) + \cot a}{x^2} \\
= \; & \lim_{x\to 0} \frac{2\cos(a+x)\sin^2 x}{x^2 \sin(a+2x)\sin(a+x)\sin(a)} \\
= \; & \lim_{x\to 0} \frac{2\cos(a+x)\frac{\sin^2 x}{x^2}}{\sin(a+2x)\sin(a+x)\sin(a)} \\
= \; & \frac{2\cos a}{\sin^3 a}.
\end{aligned}$$

(c)

$$\begin{aligned}
& \lim_{x\to 0} \frac{x^2}{\sqrt{1+x\sin x} - \sqrt{\cos x}} \\
= \; & \lim_{x\to 0} \frac{x^2}{\sqrt{1+x\sin x} - \sqrt{\cos x}} \cdot \frac{\sqrt{1+x\sin x} + \sqrt{\cos x}}{\sqrt{1+x\sin x} + \sqrt{\cos x}} \\
= \; & \lim_{x\to 0} \frac{x^2(\sqrt{1+x\sin x} + \sqrt{\cos x})}{1 + x\sin x - \cos x} \\
= \; & \lim_{x\to 0} \frac{x^2(\sqrt{1+x\sin x} + \sqrt{\cos x})}{2\sin^2\frac{x}{2} + x\sin x} \\
= \; & \lim_{x\to 0} \frac{\sqrt{1+x\sin x} + \sqrt{\cos x}}{2\frac{\sin^2\frac{x}{2}}{x^2} + \frac{\sin x}{x}} \\
= \; & \frac{4}{3}
\end{aligned}$$

(d) First note that

$$\begin{aligned}
&\frac{1-\cos x\cos 2x\cos 3x}{1-\cos x}\\
= &\frac{1-\cos x+\cos x-\cos x\cos 2x\cos 3x}{1-\cos x}\\
= &1+\cos x\frac{1-\cos 2x\cos 3x}{1-\cos x}\\
= &1+\cos x\cdot\frac{1-\cos 2x+\cos 2x-\cos 2x\cos 3x}{1-\cos 2x}\cdot\frac{1-\cos 2x}{1-\cos x}\\
= &1+\cos x\cdot\left(1+\cos 2x\cdot\frac{1-\cos 3x}{1-\cos 2x}\right)\cdot\frac{1-\cos 2x}{1-\cos x}\\
= &1+\cos x\cdot\frac{1-\cos 2x}{1-\cos x}+\cos x\cdot\cos 2x\cdot\frac{1-\cos 3x}{1-\cos x}
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{x\to 0}\frac{1-\cos nx}{1-\cos x}\\
= &\lim_{x\to 0}\frac{\sin^2\frac{nx}{2}}{\sin^2\frac{x}{2}}\\
= &\lim_{x\to 0}\frac{\sin^2\frac{nx}{2}}{\frac{n^2x^2}{2^2}}\cdot\frac{\frac{n^2x^2}{2^2}}{\sin^2\frac{x}{2}}\\
= &n^2,
\end{aligned}$$

one gets

$$\begin{aligned}
&\lim_{x\to 0}\frac{1-\cos x\cos 2x\cos 3x}{1-\cos x}\\
= &\lim_{x\to 0}\left(1+\cos x\cdot\frac{1-\cos 2x}{1-\cos x}+\cos x\cdot\cos 2x\cdot\frac{1-\cos 3x}{1-\cos x}\right)\\
= &1+2^2+3^2\\
= &14.
\end{aligned}$$

(e)

$$\begin{aligned}
&\lim_{x\to+\infty}\frac{(x+a)^{x+a}(x+b)^{x+b}}{(x+a+b)^{2x+a+b}}\\
=\ &\lim_{x\to+\infty}\frac{(1+\frac{a}{x})^{x+a}(1+\frac{b}{x})^{x+b}}{(1+\frac{a+b}{x})^{2x+a+b}}\\
=\ &\lim_{x\to+\infty}\frac{\left[(1+\frac{a}{x})^{1+\frac{x}{a}}\right]^{a}\left[(1+\frac{b}{x})^{1+\frac{x}{b}}\right]^{b}}{\left[(1+\frac{a+b}{x})^{1+\frac{2x}{a+b}}\right]^{a+b}}\\
=\ &\lim_{x\to+\infty}\frac{\left[(1+\frac{a}{x})^{\frac{x}{a}}(1+\frac{a}{x})\right]^{a}\left[(1+\frac{b}{x})^{\frac{x}{b}}(1+\frac{b}{x})\right]^{b}}{\left[(1+\frac{a+b}{x})^{\frac{2x}{a+b}}(1+\frac{a+b}{x})\right]^{a+b}}\\
=\ &\frac{e^a\cdot e^a}{e^{2(a+b)}}\\
=\ &e^{-a-b}
\end{aligned}$$

(f)

$$\begin{aligned}
\lim_{x\to0}\frac{\sqrt{1+x\sin x}-1}{e^{x^2}-1} &= \lim_{x\to0}\frac{\sqrt{1+x\sin x}-1}{x^2}\frac{x^2}{e^{x^2}-1}\\
&= \lim_{x\to0}\frac{\sqrt{1+x\sin x}-1}{x^2}\frac{\sqrt{1+x\sin x}+1}{\sqrt{1+x\sin x}+1}\frac{x^2}{e^{x^2}-1}\\
&= \lim_{x\to0}\frac{x\sin x}{x^2(\sqrt{1+x\sin x}+1)}\frac{x^2}{e^{x^2}-1}\\
&= \lim_{x\to0}\frac{\frac{\sin x}{x}}{\sqrt{1+x\sin x}+1}\frac{x^2}{e^{x^2}-1}\\
&= \frac{1}{2}
\end{aligned}$$

□

4 Continuity

4.1 Definition of continuity

Exercise 1. (Level 2)
The function f is continuous at $x = 0$ and is defined for $-1 < x < 1$ by

$$f(x) = \begin{cases} \frac{2a}{x}\ln(1+x) & \text{if } -1 < x < 0 \\ b & \text{if } x = 0 \\ \frac{x^2 \cos x}{1-\sqrt{1-x^2}} & \text{if } 0 < x < 1. \end{cases}$$

Determine the values of the constants a and b.

Solution. Consider two-sided limits:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 2a\ln(1+x)^{\frac{1}{x}} = 2a$$

and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \cos x(1+\sqrt{1-x^2}) = 2.$$

f is continuous at 0, then $2a = b = 2$, hence we get $a = 1$ and $b = 2$. □

Exercise 2. (Level 3)
Evaluate the following limits.

(a) $\lim_{x\to+\infty} \left(\sin\sqrt{x+1} - \sin\sqrt{x}\right)$

(b) $\lim_{x\to 0} \left(\frac{\cos x}{\cos 2x}\right)^{\frac{1}{x^2}}$

(c) $\lim_{x\to 0} \left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}$ $(a, b, c > 0)$

(d) $\lim_{x\to 0} \left(\frac{\cos(xe^x)-\cos(xe^{-x})}{x^3}\right)$

Solution. (a) We have

$$\begin{aligned}
& \lim_{x\to+\infty} \left(\sin\sqrt{x+1} - \sin\sqrt{x}\right) \\
= & \lim_{x\to+\infty} 2\left(\cos\frac{\sqrt{x+1}+\sqrt{x}}{2}\sin\frac{\sqrt{x+1}-\sqrt{x}}{2}\right) \\
= & \lim_{x\to+\infty} 2\left(\cos\frac{\sqrt{x+1}+\sqrt{x}}{2}\sin\left(\frac{\sqrt{x+1}-\sqrt{x}}{2}\frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}}\right)\right) \\
= & \lim_{x\to+\infty} 2\left(\cos\frac{\sqrt{x+1}+\sqrt{x}}{2}\sin\frac{1}{2(\sqrt{x+1}+\sqrt{x})}\right) \\
= & \ 0,
\end{aligned}$$

The last equality is justified by the sandwich theorem.

(b) Note that

$$\begin{aligned}
\lim_{x\to 0}(\cos nx)^{\frac{1}{x^2}} &= \lim_{x\to 0}\left(1-2\sin^2\frac{nx}{2}\right)^{\frac{1}{x^2}} \\
&= e^{\lim\limits_{x\to 0}\ln\left(1-2\sin^2\frac{nx}{2}\right)^{\frac{1}{x^2}}} \\
&= e^{\lim\limits_{x\to 0}\frac{-2\sin^2\frac{nx}{2}}{x^2}\ln\left(1-2\sin^2\frac{nx}{2}\right)^{\frac{-1}{2\sin^2\frac{nx}{2}}}} \\
&= e^{-\frac{n^2}{2}}.
\end{aligned}$$

We get $\lim\limits_{x\to 0}\left(\frac{\cos x}{\cos 2x}\right)^{\frac{1}{x^2}} = e^{-\frac{1^2}{2}}e^{\frac{2^2}{2}} = e^{\frac{3}{2}}$.

(c) By using $\lim\limits_{x\to 0}\frac{d^x-1}{x} = \ln d$ for $d>0$, one has

$$\begin{aligned}
&\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}} \\
=\ & e^{\lim\limits_{x\to 0}\ln\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}} \\
=\ & e^{\lim\limits_{x\to 0}\frac{a^x-1+b^x-1+c^x-1}{3x}\ln\left(1+\frac{a^x-1+b^x-1+c^x-1}{3}\right)^{\frac{3}{a^x-1+b^x-1+c^x-1}}} \\
=\ & e^{\frac{\ln a+\ln b+\ln c}{3}} \\
=\ & \sqrt[3]{abc}
\end{aligned}$$

(d)

$$\begin{aligned}
&\lim_{x\to 0}\left(\frac{\cos(xe^x)-\cos(xe^{-x})}{x^3}\right) \\
=\ & \lim_{x\to 0}\left(\frac{-2}{x^3}\sin(\frac{e^x+e^{-x}}{2}x)\sin(\frac{e^x-e^{-x}}{2}x)\right) \\
=\ & \lim_{x\to 0}\left(-2\frac{\sin(\frac{e^x+e^{-x}}{2}x)}{\frac{e^x+e^{-x}}{2}x}\frac{e^x+e^{-x}}{2}\frac{\sin(\frac{e^x-e^{-x}}{2}x)}{\frac{e^x-e^{-x}}{2}x}\frac{e^x-e^{-x}}{2x}\right) \\
=\ & \lim_{x\to 0}\left(-2e^{-x}\frac{\sin(\frac{e^x+e^{-x}}{2}x)}{\frac{e^x+e^{-x}}{2}x}\frac{e^x+e^{-x}}{2}\frac{\sin(\frac{e^x-e^{-x}}{2}x)}{\frac{e^x-e^{-x}}{2}x}\frac{e^{2x}-1}{2x}\right) \\
=\ & -2
\end{aligned}$$

□

4.2 Intermediate value theorem

Exercise 3. (Level 3)
Suppose $f:[0,1]\to[0,1]$ is a continuous function on $[0,1]$. Show that there exists a point $x\in[0,1]$ such that $f(x)=x$.

Solution. Since $0 \leq f(x) \leq 1$ for $x \in [0,1]$, we may assume that $f(0) > 0$ and $f(1) < 1$. Otherwise $f(0) = 0$ or $f(1) = 1$, then we are done. Let $g : [0,1] \to [0,1]$ is defined by $g(x) = f(x) - x$ for $x \in [0,1]$. Note that g is a continuous function. Since $g(0) = f(0) > 0$ and $g(1) = f(1) - 1 < 0$, by intermediate value theorem, there exists a point $x \in (0,1)$ such that $g(x) = 0$ i.e. $f(x) = x$.

□

Exercise 4. (Level 4)
Let n be a positive integer greater than 2. Let $f : [0,1] \to \mathbf{R}$ is a continuous function. Suppose $f(0) = f(1)$. Define a function $g : [0, 1 - \frac{1}{n}] \to \mathbf{R}$ by $g(x) = f(x + \frac{1}{n}) - f(x)$ for all $x \in [0, 1 - \frac{1}{n}]$.

(a) Show that $\sum\limits_{k=0}^{n-1} g(\frac{k}{n}) = 0$.

(b) Deduce that there exists a point $x \in [0, 1 - \frac{1}{n}]$ such that $f(x + \frac{1}{n}) = f(x)$.

Solution. (a) $\sum\limits_{k=0}^{n-1} g(\frac{k}{n}) = \sum\limits_{k=0}^{n-1} \big(f(\frac{k+1}{n}) - f(\frac{k}{n})\big) = f(1) - f(0) = 0$.

(b) If there exists a point $x \in \{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}\}$ such that $f(x + \frac{1}{n}) = f(x)$. Then we are done.
Now suppose that for all $k \in \{0, 1, 2, \cdots, n-1\}$, $g(\frac{k}{n}) \neq 0$. Then there exist $k_0, k_1 \in \{0, 1, 2, \cdots, n-1\}$ such that $g(\frac{k_0}{n}) > 0$ and $g(\frac{k_1}{n}) < 0$. By intermediate value theorem with the continuity of g, we get a point $x \in [0, 1 - \frac{1}{n}]$ such that $g(x) = 0$, that is $f(x + \frac{1}{n}) = f(x)$.

□

5 Differentiation

5.1 Definition of differentiability

Exercise 1. (Level 2)
Investigate the differentiability of the following functions at $x = 0$.

(a) $y = \frac{|x|}{1+x^2}$

(b) $y = \begin{cases} \tan^{-1}\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

(c) $y = \begin{cases} x\tan^{-1}\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

(d) $y = \begin{cases} x^2\tan^{-1}\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

Solution. (a) Let $f(x) = \frac{|x|}{1+x^2}$. Consider $\frac{f(0+h)-f(0)}{h}$. Note that $\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{1}{h}\frac{|h|}{1+h^2} = -1$ and $\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{1}{h}\frac{|h|}{1+h^2} = 1$, hence $f(x)$ is not differentiable at $x = 0$.

(b) The function is not continuous at $x = 0$, so it is not differentiable.

(c) Let $g(x) = \begin{cases} x\tan^{-1} x & x \neq 0 \\ 0 & x = 0 \end{cases}$.

Consider $\frac{g(0+h)-g(0)}{h}$. Note that $\lim_{h\to 0^-} \frac{g(0+h)-g(0)}{h} = \lim_{h\to 0^-} \tan^{-1}\frac{1}{h} = -\frac{\pi}{2}$ and $\lim_{h\to 0^+} \frac{g(0+h)-g(0)}{h} = \lim_{h\to 0^+} \tan^{-1}\frac{1}{h} = \frac{\pi}{2}$, hence $g(x)$ is not differentiable at $x = 0$.

(d) Let $k(x) = \begin{cases} x^2\tan^{-1}\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$.

Consider $\frac{k(0+h)-k(0)}{h}$. Note that, by the sandwich theorem $\lim_{h\to 0^-} \frac{k(0+h)-k(0)}{h} = \lim_{h\to 0^-} h\tan^{-1}\frac{1}{h} = 0$ and $\lim_{h\to 0^+} \frac{k(0+h)-k(0)}{h} = \lim_{h\to 0^+} h\tan^{-1}\frac{1}{h} = 0$, hence $k(x)$ is differentiable at $x = 0$.

□

Exercise 2. (Level 2)
Find $\frac{dy}{dx}$, from the first principles, of the following functions:

(a) $y = \frac{1}{x^2+x+1}$

(b) $y = 2x\sin 2x$

(c) $y = 6\sqrt[3]{x^4} + \frac{4}{\sqrt{x}}$

Solution. (a)

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{1}{h}\left[\frac{1}{(x+h)^2+(x+h)+1}-\frac{1}{x^2+x+1}\right]\\ &= \lim_{h\to 0}-\frac{2x+h+1}{[(x+h)^2+(x+h)+1](x^2+x+1)}\\ &= -\frac{2x+1}{(x^2+x+1)^2}\end{aligned}$$

(b)

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{2(x+h)\sin(2x+2h)-2x\sin 2x}{h}\\ &= \lim_{h\to 0}\frac{2x[\sin(2x+2h)-\sin 2x]+2h\sin(2x+2h)}{h}\\ &= \lim_{h\to 0}\frac{4x\cos(2x+h)\sin h+2h\sin(2x+2h)}{h}\\ &= \lim_{h\to 0}\left[\frac{4x\cos(2x+h)\sin h}{h}+2\sin(2x+2h)\right]\\ &= 4x\cos 2x+2\sin 2x\end{aligned}$$

(c)

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{1}{h}\left[6(x+h)^{\frac{4}{3}}+\frac{4}{(x+h)^{\frac{1}{2}}}-6x^{\frac{4}{3}}-\frac{4}{x^{\frac{1}{2}}}\right]\\ &= \lim_{h\to 0}\frac{1}{h}\left[6(x+h)^{\frac{4}{3}}-6x^{\frac{4}{3}}+\frac{4}{(x+h)^{\frac{1}{2}}}-\frac{4}{x^{\frac{1}{2}}}\right]\\ &= \lim_{h\to 0}\frac{1}{h}\left[\frac{6(x+h)^4-6x^4}{(x+h)^{\frac{8}{3}}+(x+h)^{\frac{4}{3}}x^{\frac{4}{3}}+x^{\frac{8}{3}}}+\frac{4x-4(x+h)}{x^{\frac{1}{2}}(x+h)^{\frac{1}{2}}[x^{\frac{1}{2}}+(x+h)^{\frac{1}{2}}]}\right]\\ &= \lim_{h\to 0}\frac{1}{h}\left[\frac{6(4x^3+6x62h+4xh^2+h^3)}{(x+h)^{\frac{8}{3}}+(x+h)^{\frac{4}{3}}x^{\frac{4}{3}}+x^{\frac{8}{3}}}-\frac{4}{x^{\frac{1}{2}}(x+h)^{\frac{1}{2}}[x^{\frac{1}{2}}+(x+h)^{\frac{1}{2}}]}\right]\\ &= 8x^{\frac{1}{3}}-2x^{-\frac{3}{2}}\end{aligned}$$

□

5.2 Derivatives and Chain Rule

Exercise 3. (Level 1)

Let $f:(-1,1)\to\mathbf{R}$ be a function defined by $f(x)=xe^{\sec^3 x}\sin\cos x\sin^{-1}\tan^{-1}\frac{2x^2-7x}{9}$. Find $f'(0)$.

Solution. We start by the first principle:

$$f'(0)=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\left(e^{\sec^3 h}\sin\cos h\sin^{-1}\tan^{-1}\frac{2h^2-7h}{9}\right)=0.$$

□

Exercise 4. (Level 2)
Find the derivative of the following:

(a) $y = \frac{\sin x}{\sqrt{a^2\cos^2 + b^2\sin^2 x}}$

(b) $y = x^{a^a} + a^{x^a} + a^{a^x} + x^{x^a} + x^{a^x} + a^{x^x} + x^{x^x} + a^{a^a}$

(c) $y = x10^x$

(d) $y = x^{\sin x}$

(e) $y = x(\sin x)^{\cos x}$

(f) $y = (x^2+2x-1)^{\frac{3}{2}}(x^4-3)^3$

Solution. (a)

$$\begin{aligned}
&\frac{dy}{dx} \\
&= \frac{\cos x\sqrt{a^2\cos^2 + b^2\sin^2 x} - \sin x\dfrac{-2a^2\cos x\sin x + 2b^2\sin x\cos x}{2\sqrt{a^2\cos^2 + b^2\sin^2 x}}}{a^2\cos^2 + b^2\sin^2 x} \\
&= \frac{\cos x(a^2\cos^2 + b^2\sin^2 x) + (a^2 - b^2)\cos x\sin^2 x}{(a^2\cos^2 + b^2\sin^2 x)^{\frac{3}{2}}} \\
&= \frac{a^2\cos x(\cos^2 + \sin^2 x)}{(a^2\cos^2 + b^2\sin^2 x)^{\frac{3}{2}}} \\
&= \frac{a^2\cos x}{(a^2\cos^2 + b^2\sin^2 x)^{\frac{3}{2}}}
\end{aligned}$$

(b) Note that

$$(x^x)' = (e^{x\ln x})' = (1+\ln x)x^x,$$

$$(x^{x^a})' = (e^{x^a\ln x})' = (x^{a-1} + ax^{a-1}\ln x)e^{x^a\ln x} = x^{a-1}(1+a\ln x)x^{x^a},$$

$$(x^{a^x})' = (e^{a^x\ln x})' = (\frac{a^x}{x} + \ln a a^x\ln x)e^{a^x\ln x} = a^x(\frac{1}{x} + \ln a\ln x)x^{a^x},$$

$$(a^{x^x})' = (e^{x^x\ln a})' = x^x\ln a(1+\ln x)e^{x^x\ln a} = x^x\ln a(1+\ln x)a^{x^x}$$

and

$$(x^{x^x})' = (e^{x^x\ln x})' = x^x(\ln x + (\ln x)^2 + \frac{1}{x})e^{x^x\ln x} = x^x(\ln x + (\ln x)^2 + \frac{1}{x})x^{x^x},$$

we have

$$\begin{aligned}
&\frac{dy}{dx} \\
= \; & a^a x^{a^a-1} + a\ln a x^{a-1}a^{x^a} + (\ln a)^2 a^x a^{a^x} + x^{a-1}(1+a\ln x)x^{x^a} \\
&+ a^x(\frac{1}{x} + \ln a\ln x)x^{a^x} + x^x\ln a(1+\ln x)a^{x^x} + x^x(\ln x + (\ln x)^2 + \frac{1}{x})x^{x^x}.
\end{aligned}$$

(c)

$$\frac{dy}{dx} = 10^x + x10^x \ln 10$$

(d)

$$\frac{dy}{dx} = \frac{de^{\sin x \ln x}}{dx} = (\frac{\sin x}{x} + \cos x \ln x)e^{\sin x \ln x} = (\frac{\sin x}{x} + \cos x \ln x)x^{\sin x}$$

(e) Note that

$$\begin{aligned} ((\sin x)^{\cos x})' &= \left(e^{\cos x \ln \sin x}\right)' \\ &= (\cot x \cos x - \sin x \ln \sin x)e^{\cos x \ln \sin x} \\ &= (\cot x \cos x - \sin x \ln \sin x)(\sin x)^{\cos x}, \end{aligned}$$

we have

$$\frac{dy}{dx} = x(\cot x \cos x - \sin x \ln \sin x)(\sin x)^{\cos x} + (\sin x)^{\cos x}.$$

(f) Taking the natural logarithmic

$$\ln y = \frac{3}{2}\ln(x^2 + 2x - 1) + 3\ln(x^4 - 3),$$

and differentiation give

$$\begin{aligned} \frac{dy}{dx} &= y\left(\frac{3}{2} \cdot \frac{2x+2}{x^2+2x-1} + 3 \cdot \frac{4x^3}{x^4-3}\right) \\ &= (x^2+2x-1)^{\frac{3}{2}}(x^4-3)^3\left(\frac{3x+3}{x^2+2x-1} + \frac{12x^3}{x^4-3}\right). \end{aligned}$$

□

Exercise 5. (Level 2)

If $x^y = y^x$, prove that $\frac{dy}{dx} = \frac{xy \ln y - y^2}{xy \ln x - x^2}$.

Solution. Differentiation both sides gives $x^y(\frac{y}{x} + \ln x \frac{dy}{dx}) = y^x(\ln y + \frac{x}{y}\frac{dy}{dx})$, and hence $\frac{dy}{dx} = \frac{xy \ln y - y^2}{xy \ln x - x^2}$. □

Exercise 6. (Level 3)

Using the identity

$$\cos\frac{x}{2}\cos\frac{x}{4}\cdots\cos\frac{x}{2^n} = \frac{\sin x}{2^n \sin\frac{x}{2^n}},$$

derive the sum

$$S_n = \frac{1}{2}\tan\frac{x}{2} + \frac{1}{4}\tan\frac{x}{4} + \cdots + \frac{1}{2^n}\tan\frac{x}{2^n}.$$

Solution. Taking the natural logarithmic

$$\ln\cos\frac{x}{2}+\ln\cos\frac{x}{4}+\cdots+\ln\cos\frac{x}{2^n}=\ln\sin x-\ln 2^n-\ln\sin\frac{x}{2^n},$$

and differentiation give

$$\frac{1}{2}\tan\frac{x}{2}+\frac{1}{4}\tan\frac{x}{4}+\cdots+\frac{1}{2^n}\tan\frac{x}{2^n}=\cot x-\cot\frac{x}{2^n}.$$

□

Exercise 7. (Level 3)
Find a polynomial $P(x)$ such that

$$P'(x)-2P(x)=x^n,$$

n being a given positive integer.

Solution. Let $P(x)=\sum_{r=0}^{m}a_rx^r$, then from $P'(x)-2P(x)=x^n$, we have

$$\begin{aligned}\sum_{r=1}^{m}ra_rx^{r-1}-2\sum_{r=0}^{m}a_rx^r &= x^n\\ \sum_{r=0}^{m-1}(r+1)a_{r+1}x^r-2\sum_{r=0}^{m}a_rx^r &= x^n\\ \sum_{r=0}^{n-1}(r+1)a_{r+1}x^r-2\sum_{r=0}^{n-1}a_rx^r+\sum_{r=n}^{m-1}(r+1)a_{r+1}x^r-2\sum_{r=n}^{m}a_rx^r &= x^n\end{aligned}$$

Comparing coefficients, one gives when $m\ge r\ge n+1$, then $a_r=0$. Hence we have

$$\sum_{r=0}^{n-1}(r+1)a_{r+1}x^r-2\sum_{r=0}^{n-1}a_rx^r-2a_nx^n=x^n$$

Again by comparing coefficients, it gives

$$-2a_n=1\quad\text{and}\quad(r+1)a_{r+1}-2a_r=0\quad\text{for } r=1,2,3,\cdots,n-1,$$

then

$$a_n=-\frac{1}{2}\quad\text{and}\quad a_r=\frac{r+1}{2}a_{r+1}\quad\text{for } r=1,2,3,\cdots,n-1.$$

Thus, for $r=1,2,3,\cdots,n$,

$$a_{n-r}=-\frac{1}{2}\frac{n(n-1)\cdots(n-r)}{2^r},$$

implies that

$$P(x)=-\frac{1}{2}\left[x^n+\frac{n}{2}x^{n-1}+\cdots+\frac{n!}{2^n}\right].$$

□

5.3 Higher derivatives (Leibniz's Rule)

Exercise 8. (Level 3)
Let $f(x) = x^n e^x$ where n is a positive integer.

(a) Prove that

$$e^{-x} f^{(n)}(x) = \sum_{r=0}^{n} \{\frac{n!}{(n-r)!}\}^2 \frac{x^{n-r}}{r!}.$$

(b) Evaluate $f^{(2n)}(0)$.

Solution. (a) By Leibniz theorem,

$$\begin{aligned}\frac{d^n}{dx^n}(x^n e^x) &= \sum_{r=0}^{n} C_r^n (x^n)^{(r)} (e^x)^{(n-r)} \\ &= \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} n(n-1)\cdots(n-r+1)x^{n-r}e^x \\ &= \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} \frac{n!}{(n-r)!} x^{n-r}e^x\end{aligned}$$

which gives the result.

(b) By Leibniz theorem,

$$\begin{aligned}\frac{d^{2n}}{dx^{2n}}(x^n e^x) &= \sum_{r=0}^{2n} C_r^{2n} (x^n)^{(r)} (e^x)^{(2n-r)} \\ &= \sum_{r=0}^{n} \frac{(2n)!}{r!(2n-r)!} n(n-1)\cdots(n-r+1)x^{n-r}e^x\end{aligned}$$

which gives $f^{(2n)}(0) = \frac{(2n)!}{n!}$.

□

Exercise 9. (Level 4)
Let $y = (\sin^{-1} x)^2$

(a) Prove that

(i) $(1-x^2)y'' - xy' = 2$,

(ii) $(1-x^2)y^{(n+2)} - (2n+1)xy^{(n+1)} - n^2 y^{(n)} = 0$.

(b) Deduce that $y^{(n+2)}(0) = n^2 y^{(n)}(0)$ and that

$$y^{(2n)}(0) = 2[(2n-2)(2n-4)\cdots 4\cdot 2]^2 \quad \text{and} \quad y^{(2n+1)}(0) = 0.$$

Solution. (a) Prove that

(i) $y' = \frac{2\sin^{-1}x}{\sqrt{1-x^2}}$ gives $\sqrt{1-x^2}y' = 2\sin^{-1}x$. Differentiation gives $\sqrt{1-x^2}y'' - \frac{x}{\sqrt{1-x^2}}y' = \frac{2}{\sqrt{1-x^2}}$, that is, $(1-x^2)y'' - xy' = 2$.

(ii) Differentiate n times by Leibniz theorem, we have that

$$\begin{aligned} 0 &= ((1-x^2)y'' - xy')^{(n)} \\ &= (1-x^2)y^{(n+2)} + n(-2x)y^{(n+1)} + \frac{n(n-1)}{2}(-2)y^{(n)} - xy^{(n+1)} - ny^{(n)} \\ &= (1-x^2)y^{(n+2)} - (2n+1)xy^{(n+1)} - n^2y^{(n)} \end{aligned}$$

(b) Putting $x = 0$ in (a)(ii), we get that $y^{(n+2)}(0) = n^2y^{(n)}(0)$.

By the preceding equation and $y'(0) = \frac{2\sin^{-1}0}{\sqrt{1-0^2}} = 0$, we have

$$y^{(2n+1)}(0) = y^{(2n-1)}(0) = \cdots = y'(0) = 0.$$

Again, by $y''(0) = 2$, one obtains

$$\begin{aligned} y^{(2n)}(0) &= (2n-2)^2y^{(2n-2)}(0) \\ &\vdots \\ &= [(2n-2)(2n-4)\cdots 4\cdot 2]^2y''(0) \\ &= 2[(2n-2)(2n-4)\cdots 4\cdot 2]^2. \end{aligned}$$

□

6 Application of differentiation

6.1 Rolle's Theorem

Exercise 1. (Level 3)
Prove that the equation

$$a_1x + a_2x^2 + \cdots + a_nx^n = \frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1}$$

is solvable.

Solution. Let $f; \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = \frac{a_1x^2}{2} + \frac{a_2x^3}{3} + \cdots + \frac{a_nx^n}{n+1} - \left(\frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1}\right)x$$

for all $x \in \mathbf{R}$. Note that f is differentiable on $\mathbf{R}$ and that $f'(x) = a_1x + a_2x^2 + \cdots + a_nx^n - \left(\frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1}\right)$. For $f(0) = f(1) = 0$ therefore, by Rolle's theorem, there is an $c \in (0,1)$ such that $f'(c) = 0$ which shows the result.

□

Exercise 2. (Level 3)
Let f be a smooth function. Prove that, if $a < c < b$, there is an $\zeta \in (a,b)$ such that

$$\frac{f(a)}{(a-b)(a-c)} + \frac{f(b)}{(b-c)(b-a)} + \frac{f(c)}{(c-b)(c-a)} = \frac{f''(\zeta)}{2}.$$

(Hint: Consider $F(x) = f(x) - \frac{(x-b)(x-c)}{(a-b)(a-c)}f(a) + \frac{(x-c)(x-a)}{(b-c)(b-a)}f(b) + \frac{(x-b)(x-a)}{(c-b)(c-a)}f(c)$.)

Solution. Let $F(x) = f(x) - \frac{(x-b)(x-c)}{(a-b)(a-c)}f(a) + \frac{(x-c)(x-a)}{(b-c)(b-a)}f(b) + \frac{(x-b)(x-a)}{(c-b)(c-a)}f(c)$. Also F is smooth. Since $F(a) = F(b) = F(c) = 0$, by Rolle's theorem, there exist $\xi \in (a,c)$ and $\eta \in (b,c)$ such that $F(\xi) = F(\eta) = 0$. Again, by Rolle's theorem, there is an $\zeta \in (\xi, \eta) \subset (a,b)$ such that $F''(\zeta) = 0$. This shows the result.

□

6.2 Lagrange's mean value theorem and Cauchy's mean value theorem

Exercise 3. (Level 1)
Suppose a function f is continuous on $[a,b]$ and differentiable on (a,b). Decide whether it is possible to find for each point ξ of (a,b), there are two points c, d, $a \le c < \xi < d \le b$, such that

$$\frac{f(c) - f(d)}{c - d} = f'(\xi).$$

Solution. It is impossible. Suppose not, we consider $f(x) = x^3$ and $\xi = 0$. Then c must be negative while d must be positive, and

$$0 = 3(0)^2 = \frac{f(c) - f(d)}{c - d} = \frac{c^3 - d^3}{c - d} > 0$$

which is absurd.

□

Exercise 4. (Level 2)
Show that for $x \in (0, \frac{\pi}{2})$, $f(x) = \sin x \tan x - 2\ln\sec x$ is always positive.

Solution. Differentiation gives

$$f'(x) = \cos x \tan x + \sin x \sec^2 x - \frac{2\sec x \tan x}{\sec x} = \sin x(\sec x - 1)^2 > 0,$$

therefore $f(x) > f(0) = 0$ for all $x \in (0, \frac{\pi}{2})$.

□

Exercise 5. (Level 3)
Let I be an open interval and $a, b \in I$ with $a < b$. If f is differentiable on I and if λ is a number between $f'(a)$ and $f'(b)$, show that there is at least one point $c \in (a, b)$ such that $f'(c) = \lambda$.

(Hint: You may start with defining a function $f_a(t) = \begin{cases} f'(a) & \text{if } t = a \\ \frac{f(a)-f(t)}{t-a} & \text{if } t \neq a \end{cases}$.

Solution. Define functions $f_a, f_b : [a, b] \to \mathbf{R}$ by

$$f_a(t) = \begin{cases} f'(a) & \text{if } t = a \\ \frac{f(a)-f(t)}{t-a} & \text{if } t \neq a \end{cases}$$

and

$$f_b(t) = \begin{cases} f'(b) & \text{if } t = b \\ \frac{f(b)-f(t)}{t-b} & \text{if } t \neq b \end{cases}.$$

Since f is differentiable at $x = a$, $x = b$, f_a and f_b are continuous. From the definitions, it follows that $f_a(a) = f'(a)$, $f_a(b) = f_b(a)$ and $f_b(b) = f'(b)$. Hence λ lies between $f_a(a)$ and $f_a(b)$ or λ lies between $f_b(a)$ and $f_b(b)$. Without loss of generality, we assume λ lies between $f_a(a)$ and $f_a(b)$. By the intermediate value theorem, there exists s in (a, b) such that

$$\lambda = f_a(s) = \frac{f(a) - f(s)}{a - s}.$$

Thanks the mean value theorem, there is an $c \in (a, s)$ such that

$$f'(c) = \frac{f(a) - f(s)}{a - s} = \lambda.$$

□

Exercise 6. (Level 3/Level 4)
Show that, when $0 < \theta < 2\pi$, $1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} > \cos\theta > 1 - \frac{\theta^2}{2}$.

Solution. Let $f(\theta) = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \cos\theta$. Note that for $0 < \theta < 2\pi$,

$$\begin{aligned} f'(\theta) &= -\theta + \frac{\theta^3}{6} + \sin\theta, \\ f''(\theta) &= -1 + \frac{\theta^2}{3} + \cos\theta, \\ f^{(3)}(\theta) &= \theta - \sin\theta, \\ f^{(4)}(\theta) &= 1 - \cos\theta > 0, \end{aligned}$$

and

$$f(0) = f'(0) = f''(0) = f^{(3)}(0) = f^{(4)}(0) = 0.$$

It follows that $f^{(3)}$ is increasing on $[0, 2\pi]$, and hence $f^{(3)}(\theta) > f^{(3)}(0) = 0$ for $0 < \theta < 2\pi$. Apply the argument in the same manner, we get $f^{''}(\theta) > 0$, $f^{'}(\theta) > 0$ and $f(\theta) > 0$ for $0 < \theta < 2\pi$. Then one gets when $0 < \theta < 2\pi$, $1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} > \cos\theta$.

Similarly, let $g(\theta) = \cos\theta - 1 + \frac{\theta^2}{2}$. Note that for $0 < \theta < 2\pi$,

$$\begin{aligned} g'(\theta) &= -\sin\theta + \theta, \\ g''(\theta) &= 1 - \cos\theta > 0, \end{aligned}$$

and

$$g(0) = g'(0) = 0.$$

We get when $0 < \theta < 2\pi$, $\cos\theta > 1 - \frac{\theta^2}{2}$, so $1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} > \cos\theta > 1 - \frac{\theta^2}{2}$ is established. □

Exercise 7. (Level 5)

(a) Prove that for $0 < \theta < \frac{\pi}{2}$, $\tan\theta > \theta > \sin\theta$.

(b) (i) Show that $C_3^{2n-1} - 2C_1^{2n-1} - 2C_3^{2n+1} - 2C_1^{2n+1} = -C_3^{2n+3}$.

(ii) Show that $2C_{2r+1}^{2n+1} + 2C_{2r-1}^{2n+1} - C_{2r+1}^{2n-1} + 2C_{2r-1}^{2n-1} - C_{2r-3}^{2n-1} = C_{2r+1}^{2n+3}$ for $r \in \{2, 3, \cdots, n-1\}$.

(iii) Prove that for $n \in \mathbf{N}$,

$$\sin(2n+1)\theta = \sin^{2n+1}\theta \sum_{r=0}^{n} (-1)^r C_{2r+1}^{2n+1} (\cot^2\theta)^{n-r},$$

where $0 < \theta < \frac{\pi}{2}$.

(c) (i) Given that $\alpha_1, \cdots, \alpha_n$ are the roots of $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ where $a_n \neq 0$.

A. Show that $\alpha_1 + \cdots + \alpha_n = -\frac{a_{n-1}}{a_n}$.

B. Furthermore, $\alpha_1, \cdots, \alpha_n$ are pairwise distinct, does there exist a $\beta \notin \{\alpha_1, \cdots, \alpha_n\}$ such that $a_n\beta^n + a_{n-1}\beta^{n-1} + \cdots + a_0 = 0$?

(ii) Using (b), show that

$$\sum_{k=1}^{n} \cot^2 \frac{k\pi}{2n+1} = \frac{n(2n-1)}{3},$$

and deduce that

$$\sum_{k=1}^{n} \csc^2 \frac{k\pi}{2n+1} = \frac{n(2n+2)}{3}.$$

(d) Let $A_n = \sum\limits_{k=1}^{n} \frac{1}{k^2}$. Show that

$$\frac{\pi^2}{6}\left(\frac{2n}{2n+1}\right)\left(\frac{2n-1}{2n+1}\right) < A_n < \frac{\pi^2}{6}\left(\frac{2n}{2n+1}\right)\left(\frac{2n+2}{2n+1}\right).$$

Hence evaluate $\lim\limits_{n\to+\infty} A_n$.

Solution. (a) Let $f(\theta) = \tan\theta - \theta$ and $f(\theta) = \sin\theta - \theta$ for $0 < \theta < \frac{\pi}{2}$. Differentiation gives $f'(\theta) = \sec^2\theta - 1 > 0$ and $g'(\theta) = \cos\theta - 1 < 0$. Then f is strictly increasing and g is strictly decreasing. Thus, for $0 < \theta < \frac{\pi}{2}$, $f(\theta) > f(0)$ and $g(\theta) < g(0)$, i.e. $\tan\theta > \theta > \sin\theta$.

(b) (i) Compute in a straightforward way.

(ii) Using the fact that $C_r^n + C_{r-1}^n = C_r^{n+1}$ repeatedly, one obtains for $r \in \{2, 3, \cdots, n-1\}$

$$\begin{aligned} C_{2r+1}^{2n+3} &= C_{2r+1}^{2n+2} + C_{2r}^{2n+2} \\ &= C_{2r+1}^{2n+1} + 2C_{2r}^{2n+1} + C_{2r-1}^{2n+1} \end{aligned}$$

and

$$\begin{aligned} & C_{2r+1}^{2n+1} - 2C_{2r}^{2n+1} + C_{2r-1}^{2n+1} \\ = \; & C_{2r+1}^{2n+1} - C_{2r}^{2n+1} + C_{2r-1}^{2n+1} - C_{2r}^{2n+1} \\ = \; & C_{2r+1}^{2n} + C_{2r}^{2n} - C_{2r}^{2n} - C_{2r-1}^{2n} + C_{2r-1}^{2n} + C_{2r-2}^{2n} - C_{2r}^{2n} - C_{2r-1}^{2n} \\ = \; & C_{2r+1}^{2n} - C_{2r-1}^{2n} + C_{2r-2}^{2n} - C_{2r}^{2n} \\ = \; & C_{2r+1}^{2n-1} + C_{2r}^{2n-1} - C_{2r-1}^{2n-1} - C_{2r-2}^{2n-1} + C_{2r-2}^{2n-1} + C_{2r-3}^{2n-1} - C_{2r}^{2n-1} - C_{2r-1}^{2n-1} \\ = \; & C_{2r+1}^{2n-1} - 2C_{2r-1}^{2n-1} + C_{2r-3}^{2n-1}. \end{aligned}$$

Combining, we have for $r \in \{2, 3, \cdots, n-1\}$

$$\begin{aligned} & 2C_{2r+1}^{2n+1} + 2C_{2r-1}^{2n+1} - C_{2r+1}^{2n-1} + 2C_{2r-1}^{2n-1} - C_{2r-3}^{2n-1} \\ = \; & C_{2r+1}^{2n+1} + 2C_{2r}^{2n+1} + C_{2r-1}^{2n+1} \\ = \; & C_{2r+1}^{2n+3}. \end{aligned}$$

(iii) For $n = 1$, we have

$$\begin{aligned} \sin 3\theta &= \sin(2\theta + \theta) \\ &= \sin 2\theta \cos\theta + \sin\theta \cos 2\theta \\ &= 2\sin\theta\cos^2\theta + \sin\theta(\cos^2\theta - \sin^2\theta) \\ &= 3\sin\theta\cos^2\theta - \sin^3\theta \\ &= \sin^3\theta(3\cot^2\theta - 1) \\ &= \sin^3\theta \sum_{r=0}^{n} (-1)^r C_{2r+1}^3 (\cot^2\theta)^{1-r}. \end{aligned}$$

Suppose that

$$\sin(2k+1)\theta = \sin^{2k+1}\theta \sum_{r=0}^{k} (-1)^r C_{2r+1}^{2k+1} (\cot^2\theta)^{k-r},$$

for all $k \leq n$. Noting that $\sin(2n+1)\theta\cos 2\theta = \frac{1}{2}(\sin(2n+3)\theta + \sin(2n-1)\theta)$. It gives that

$$
\begin{aligned}
& \sin(2n+3)\theta \\
= \; & -\sin(2n-1)\theta + 2\sin(2n+1)\theta\cos 2\theta \\
= \; & \sin^{2n-1}\theta \sum_{r=0}^{n-1}(-1)^{r+1}C_{2r+1}^{2n-1}(\cot^2\theta)^{n-1-r} \\
& + \sin^{2n+1}\theta \sum_{r=0}^{n}(-1)^{r}C_{2r+1}^{2n+1} 2\cos 2\theta(\cot^2\theta)^{n-r} \\
= \; & \sin^{2n+3}\theta \sum_{r=0}^{n-1}(-1)^{r+1}C_{2r+1}^{2n-1}(\cot^2\theta)^{n-1-r}\csc^4\theta \\
& + \sin^{2n+3}\theta \sum_{r=0}^{n}(-1)^{r}C_{2r+1}^{2n+1}\frac{2\cos 2\theta}{\sin^2\theta}(\cot^2\theta)^{n-r} \\
= \; & \sin^{2n+3}\theta \sum_{r=0}^{n-1}(-1)^{r+1}C_{2r+1}^{2n-1}(\cot^2\theta)^{n-1-r}(\cot^4\theta + 2\cot^2\theta + 1) \\
& + \sin^{2n+3}\theta \sum_{r=0}^{n}(-1)^{r}C_{2r+1}^{2n+1}(2\cot^2\theta - 2)(\cot^2\theta)^{n-r}.
\end{aligned}
$$

The first and second summations would be treated as follows:

$$
\begin{aligned}
& \sin^{2n+3}\theta \sum_{r=0}^{n-1}(-1)^{r+1}C_{2r+1}^{2n-1}(\cot^2\theta)^{n-1-r}(\cot^4\theta + 2\cot^2\theta + 1) \\
= \; & \sin^{2n+3}\theta \sum_{r=0}^{n-1}(-1)^{r+1}C_{2r+1}^{2n-1}(\cot^2\theta)^{n+1-r} \\
& + \sin^{2n+3}\theta \sum_{r=0}^{n-1}(-1)^{r+1}2C_{2r+1}^{2n-1}(\cot^2\theta)^{n-r} \\
& + \sin^{2n+3}\theta \sum_{r=0}^{n-1}(-1)^{r+1}C_{2r+1}^{2n-1}(\cot^2\theta)^{n-1-r} \\
= \; & \sin^{2n+3}\theta \sum_{r=0}^{n-1}(-1)^{r+1}C_{2r+1}^{2n-1}(\cot^2\theta)^{n+1-r} \\
& + \sin^{2n+3}\theta \sum_{r=1}^{n}(-1)^{r}2C_{2r-1}^{2n-1}(\cot^2\theta)^{n+1-r} \\
& + \sin^{2n+3}\theta \sum_{r=2}^{n+1}(-1)^{r+1}C_{2r-3}^{2n-1}(\cot^2\theta)^{n+1-r},
\end{aligned}
$$

$$
\begin{aligned}
& \sin^{2n+3}\theta \sum_{r=0}^{n} (-1)^r C_{2r+1}^{2n+1} (2\cot^2\theta - 2)(\cot^2\theta)^{n-r} \\
= \quad & \sin^{2n+3}\theta \sum_{r=0}^{n} (-1)^r 2C_{2r+1}^{2n+1} (\cot^2\theta)^{n+1-r} \\
& + \sin^{2n+3}\theta \sum_{r=0}^{n} (-1)^r 2C_{2r+1}^{2n+1} (\cot^2\theta)^{n-r} \\
= \quad & \sin^{2n+3}\theta \sum_{r=0}^{n} (-1)^r 2C_{2r+1}^{2n+1} (\cot^2\theta)^{n+1-r} \\
& + \sin^{2n+3}\theta \sum_{r=1}^{n+1} (-1)^{r+1} 2C_{2r-1}^{2n+1} (\cot^2\theta)^{n+1-r}.
\end{aligned}
$$

Add up, by the above results, one arrives

$$\begin{aligned}
& \sin(2n+3)\theta \\
= \;& \sin^{2n+3}\theta \sum_{r=0}^{n-1} (-1)^{r+1} C_{2r+1}^{2n-1} (\cot^2\theta)^{n-1-r} (\cot^4\theta + 2\cot^2\theta + 1) \\
& + \sin^{2n+3}\theta \sum_{r=0}^{n} (-1)^{r} C_{2r+1}^{2n+1} (2\cot^2\theta - 2)(\cot^2\theta)^{n-r} \\
= \;& \sin^{2n+3}\theta \sum_{r=0}^{n-1} (-1)^{r+1} C_{2r+1}^{2n-1} (\cot^2\theta)^{n+1-r} \\
& + \sin^{2n+3}\theta \sum_{r=1}^{n} (-1)^{r} 2C_{2r-1}^{2n-1} (\cot^2\theta)^{n+1-r} \\
& + \sin^{2n+3}\theta \sum_{r=2}^{n+1} (-1)^{r+1} C_{2r-3}^{2n-1} (\cot^2\theta)^{n+1-r} \\
& + \sin^{2n+3}\theta \sum_{r=0}^{n} (-1)^{r} 2C_{2r+1}^{2n+1} (\cot^2\theta)^{n+1-r} \\
& + \sin^{2n+3}\theta \sum_{r=1}^{n+1} (-1)^{r+1} 2C_{2r-1}^{2n+1} (\cot^2\theta)^{n+1-r} \\
= \;& \sin^{2n+3}\theta (-C_1^{2n-1} + 2C_1^{2n+1})(\cot^2\theta)^{n+1} \\
& + \sin^{2n+3}\theta (C_3^{2n-1} - 2C_1^{2n-1} - 2C_3^{2n+1} - 2C_1^{2n+1})(\cot^2\theta)^{n} \\
& + \sin^{2n+3}\theta \sum_{r=2}^{n-1} (-1)^{r} (2C_{2r+1}^{2n+1} + 2C_{2r-1}^{2n+1} - C_{2r+1}^{2n-1} + 2C_{2r-1}^{2n-1} - C_{2r-3}^{2n-1})(\cot^2\theta)^{n+1-r} \\
& + \sin^{2n+3}\theta (-1)^{n} (2C_{2n-1}^{2n-1} - C_{2n-3}^{2n-1} + 2C_{2n+1}^{2n+1} + 2C_{2n-1}^{2n+1}) \cot^2\theta \\
& + \sin^{2n+3}\theta (-1)^{n+1} (-C_{2n-1}^{2n-1} + 2C_{2n+1}^{2n+1}) \\
= \;& \sin^{2n+3}\theta (\cot^2\theta)^{n+1} + \sin^{2n+3}\theta(-C_3^{2n+3})(\cot^2\theta)^{n} \\
& + \sin^{2n+3}\theta \sum_{r=2}^{n-1} (-1)^{r} (C_{2r+1}^{2n+3})(\cot^2\theta)^{n+1-r} + \sin^{2n+3}\theta(-1)^{n}(C_{2n+1}^{2n+3})\cot^2\theta \\
& + \sin^{2n+3}\theta(-1)^{n+1} \\
= \;& \sin^{2n+3}\theta \sum_{r=0}^{n+1} (-1)^{r} C_{2r+1}^{2n+3} (\cot^2\theta)^{n+1-r}.
\end{aligned}$$

Hence, we are done by mathematical induction.

(c) (i) A. Note that

$$\begin{aligned}
x^n + \frac{a_{n-1}}{a_n} x^{n-1} + \cdots + \frac{a_0}{a_n} &= (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n) \\
&= x^n - (\alpha_1 + \cdots + \alpha_n)x^{n-1} + \cdots + \alpha_1 \cdots \alpha_n
\end{aligned}$$

The result follows by comparing the coefficients of x^{n-1}

B. No. Otherwise $0 = a_n\beta^n + a_{n-1}\beta^{n-1} + \cdots + a_0 = a_n(\beta-\alpha_1)\cdots(\beta-\alpha_n)$ which implies that $a_n = 0$. It contradicts the given condition.

(ii) From (b), we may observe that $\cot^2\frac{k\pi}{2n+1}$ for $k = 1, 2, \cdots, n$ are distinct roots of $\sum_{r=0}^{n}(-1)^r C_{2r+1}^{2n+1} x^{n-r} = 0$. By A. of (i), we get

$$\sum_{k=1}^{n}\cot^2\frac{k\pi}{2n+1} = \frac{C_3^{2n+1}}{C_1^{2n+1}} = \frac{n(2n-1)}{3},$$

and hence

$$\sum_{k=1}^{n}\csc^2\frac{k\pi}{2n+1} = \sum_{k=1}^{n}\left(1+\cot^2\frac{k\pi}{2n+1}\right) = n + \frac{n(2n-1)}{3} = \frac{n(2n+2)}{3}.$$

(d) By (a), for $k \in \{1, 2, \cdots, n\}$, we get

$$\cot^2\frac{k\pi}{2n+1} < \left(\frac{k\pi}{2n+1}\right)^2 < \csc^2\frac{k\pi}{2n+1}.$$

Summing k from 1 to n, one obtains

$$\sum_{k=1}^{n}\cot^2\frac{k\pi}{2n+1} < \sum_{k=1}^{n}\left(\frac{2n+1}{k\pi}\right)^2 < \sum_{k=1}^{n}\csc^2\frac{k\pi}{2n+1}.$$

Applying the results of (c)(ii) and multiplying the constant $\frac{\pi^2}{(2n+1)^2}$, we have

$$\frac{\pi^2}{6}\left(\frac{2n}{2n+1}\right)\left(\frac{2n-1}{2n+1}\right) < A_n < \frac{\pi^2}{6}\left(\frac{2n}{2n+1}\right)\left(\frac{2n+2}{2n+1}\right).$$

Since $\lim_{n\to+\infty}\left(\frac{2n}{2n+1}\cdot\frac{2n-1}{2n+1}\right) = 1$ and $\lim_{n\to+\infty}\left(\frac{2n}{2n+1}\cdot\frac{2n+2}{2n+1}\right) = 1$, by the sandwich theorem, one gets $\lim_{n\to+\infty} A_n = \frac{\pi^2}{6}$.

□

6.3 First and second derivative check

Exercise 8. (Level 3)
Prove that the function

$$f(x) = \frac{9}{x} + \frac{1}{4-x}$$

has a maximum point and a minimum point.

Solution. Note that

$$f'(x) = -8\frac{(x-6)(x-3)}{x^2(4-x)^2} \quad \text{and} \quad f''(x) = \frac{18}{x^3} + \frac{2}{(4-x)^3}.$$

The critical values are $x = 0, 3, 4, 6$. f' does not change sign through $x = 0$ and $x = 4$, so $x = 0$ and $x = 4$ do not serve as an extremum point. (Or, you may say $x = 0$ and $x = 4$ are not any candidate for extremum point, since the domain of function exclude those two points.) Because $f''(6) < 0$ and $f''(3) > 0$, by the second derivative test, the maximum point is $(6, f(6))$ and the minimum point is $(3, f(3))$.

□

Exercise 9. (Level 5)

Let $I = [2, 3]$ and $g(x) = \frac{1}{2}\left(x + \frac{5}{x}\right)$, where $x \in I$. Let $x_0 \in I$ and define $x_{n+1} = g(x_n)$ for $n = 1, 2, 3, \cdots$.

(a) Show that the equation $x = g(x)$ has exactly one root in I.

(b) Show that $x_n \in I$ for $n = 1, 2, 3, \cdots$.

(c) Show that $|g'(x)| \le \frac{2}{9}$ for all $x \in I$.

(d) Let α be the root of $g(x) = x$ mentioned in (a).

(i) Show that $|x_n - \alpha| \le \frac{2}{9}|x_{n-1} - \alpha|$.

(ii) Show that $\{x_n\}$ converges and find the limit.

Solution. (a) Let $f(x) = x - g(x)$. Since $f(2)f(3) < 0$, by intermediate value theorem, $f(x) = 0$ i.e. $x = g(x)$ has at least one root in I. Note that $f'(x) = \frac{1}{2} + \frac{5}{2x^2} > 0$ for all $x \in I$. It says that f is a strictly increasing, and the root in I is unique.

(b) We shall investigate the absolute maximum and minimum of g on I. The derivative $g'(x) = \frac{1}{2}\left(1 - \frac{5}{x^2}\right)$. From $g'(x) = 0$, we have $x = \sqrt{5}$ or $-\sqrt{5}$. We look at $x = \sqrt{5}$. Note that

$$g''(\sqrt{5}) = \frac{5}{(\sqrt{5})^3} > 0,$$

by the second derivative test, $g(x)$ attains the local minimum at $x = \sqrt{5}$. Comparing the following three candidates:

$$g(2) = \frac{9}{4}, g(\sqrt{5}) = \sqrt{5} \quad \text{and} \quad g(3) = \frac{7}{3},$$

one obtains $\sqrt{5} = g(\sqrt{5}) \le g(x) \le g(3) = \frac{7}{3}$, and hence $2 < g(x) < 3$ for all $x \in I$. From $x_n = g(x_{n-1})$, we have $2 < x_n < 3$.

(c) For $g'(x) = \frac{1}{2}\left(1 - \frac{5}{x^2}\right)$ and $g''(x) = \frac{5}{x^3} > 0$ for all $x \in I$, so g' is increasing, therefore,

$$-\frac{1}{8} = g'(2) \le g'(x) \le g'(3) = \frac{2}{9}.$$

Then, we get $|g'(x)| \le \frac{2}{9}$ for all $x \in I$.

(d) (i) By the mean value theorem, there exists $\xi \in (2, 3)$ such that

$$\frac{g(x_{n-1}) - g(\alpha)}{x_{n-1} - \alpha} = g'(\xi).$$

Then, by (c), we have

$$|x_n - \alpha| = |g'(\xi)||x_{n-1} - \alpha| \le \frac{2}{9}|x_{n-1} - \alpha|.$$

(ii) Inductively, we get

$$|x_n - \alpha| \le \frac{2}{9}|x_{n-1} - \alpha| \cdots \le \left(\frac{2}{9}\right)^n |x_0 - \alpha|.$$

Because $\lim_{n\to\infty} \left(\frac{2}{9}\right)^n |x_0 - \alpha| = 0$, the sandwich theorem tells us that $\lim_{n\to\infty} |x_n - \alpha| = 0$. It implies, by the sandwich theorem with the inequalities $-|x_n - \alpha| \le x_n - \alpha \le |x_n - \alpha|$, that $\lim_{n\to\infty} x_n - \alpha = 0$, i.e. $\lim_{n\to\infty} x_n = \alpha$. The sequence $\{x_n\}$ converges. From $x_{n+1} = g(x_n) = \frac{1}{2}\left(x_n + \frac{5}{x_n}\right)$, we have

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(x_n + \frac{5}{x_n}\right).$$

One arrives $\alpha = \frac{1}{2}\left(\alpha + \frac{5}{\alpha}\right)$ which gives $\alpha^2 = 5$. Since $\alpha \in [2, 3]$, then we have $\alpha = \sqrt{5}$, in other words, $\lim_{n\to\infty} x_n = \sqrt{5}$.

□

6.4 Curve sketching

Exercise 10. (Level 4)
Let $f(x) = (1 + x)e^{-2x}$ be a real values function defined on $\mathbf{R}$.

(a) Find $f'(x)$ and $f''(x)$.

(b) Find, if any, the relative maximum and minimum points and the point of inflexion of $f(x)$.

(c) Find, if any, the equation of asymptote.

(d) Sketch the graph of $y = f(x)$.

(e) Sketch the graph of $y^2 = f(x)$.

(f) Sketch the graph of $y = |f(x)|$.

(g) Sketch the graph of $y = f(|x|)$.

(h) Sketch the graph of $y = f(-|x|)$.

(i) Sketch the graph of $y = f(3x + 5)$.

Solution. (a) Differentiation gives $f'(x) = -(2x + 1)e^{-2x}$ and $f''(x) = 4xe^{-2x}$.

(b) $f(x) = 0$ gives $x = -\frac{1}{2}$. Because $f''(-\frac{1}{2}) < 0$, by the second derivative test, the maximum point is $(-\frac{1}{2}, f(-\frac{1}{2})) = (-\frac{1}{2}, \frac{e}{2})$.
Note that

$$f''(x) \begin{cases} < 0 & \text{if } x < 0 \\ = 0 & \text{if } x = 0 \\ > 0 & \text{if } x > 0 \end{cases}.$$

Therefore the point of inflexion is $(0, f(0)) = (0, 1)$.

(c) First notice that $\lim_{x\to-\infty} f(x) = \infty$. Making use of the mean value theorem, we get $e^x > 1 + x$ for $x > 0$, and hence $0 < (1 + x)e^{-2x} < e^{-x}$ for $x > 0$. Then, by the sandwich theorem, we have $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \left[(1 + x)e^{-2x}\right] = 0$. Therefore, $y = 0$ is the equation of asymptote.

(d)

(e)

(f)

(g)

(h)

(i)

□

7 Indefinite integration

7.1 Primitive functions

Exercise 1. (Level 2)
Find $\int f(x)dx$ for the following functions $f(x)$.

(a) $f(x) = \pi x$

(b) $f(x) = x^{\pi}$

(c) $f(x) = x^{-\pi}$

(d) $f(x) = \sqrt[\pi]{x}$

(e) $f(x) = x^{-\frac{1}{\pi}}$

(f) $f(x) = e^{\pi}$

(g) $f(x) = \pi^{x}$

(h) $f(x) = \ln x \log_x \pi$

(i) $f(x) = \log_{\pi} x$

Solution. (a) $\int f(x)dx = \frac{\pi}{2}x^2 + C$

(b) $\int f(x)dx = \frac{x^{\pi+1}}{\pi+1} + C$

(c) $\int f(x)dx = \frac{x^{-\pi+1}}{-\pi+1} + C$

(d) $\int f(x)dx = \frac{\pi}{\pi+1}x^{\frac{\pi+1}{\pi}} + C$

(e) $\int f(x)dx = \frac{\pi}{1-\pi}x^{\frac{\pi-1}{\pi}} + C$

(f) $\int f(x)dx = e^{\pi}x + C$

(g) $\int f(x)dx = \int e^{x\ln\pi}dx = \frac{e^{x\ln\pi}}{\ln\pi} + C = \frac{\pi^x}{\ln\pi} + C$

(h) $\int f(x)dx = \int \ln x \frac{\ln\pi}{\ln x}dx = x\ln\pi + C$

(i) $\int f(x)dx = \int \frac{\ln x}{\ln\pi}dx = \frac{1}{x\ln\pi} + C$

□

Exercise 2. (Level 2)

Let $|x| < 1$. Show that the following three functions $\cos^{-1} x$, $-\sin^{-1} x$ and $2\cos^{-1}\sqrt{\frac{1}{2}(x+1)}$ are the primitive functions of a same function.

Solution. Differentiation gives
$\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$, $\frac{d}{dx}(-\sin^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$ and

$$\frac{d}{dx}\left(2\cos^{-1}\sqrt{\frac{1}{2}(x+1)}\right) = -2\frac{\frac{1}{2\sqrt{2}\sqrt{x+1}}}{\sqrt{1-\frac{1}{2}(x+1)}} = -\frac{1}{\sqrt{1-x^2}}$$

. Thus they are the primitive functions of a same function.

□

Exercise 3. (Level 3)
Find $\int f(x)dx$ for the following functions $f(x)$.

(a) $f(x) = \begin{cases} (x+2)^2 & \text{for } x > 0 \\ 2 & \text{for } x = 0 \\ 1 & \text{for } x < 0 \end{cases}$

(b) $f(x) = |x^2 + x + 6|$

(c) $f(x) = |x^2 + x - 6|$

(d) $f(x) = |x+3| + |x-2|$

(e) $f(x) = \frac{1}{|x+3|+|x-2|}$

Solution. (a) $\int f(x)dx = F(x) + C$ where $F(x) = \begin{cases} \frac{(x+2)^3}{3} & \text{for } x \geq 0 \\ x + \frac{8}{3} & \text{for } x < 0 \end{cases}$

(b) $\int f(x)dx = \frac{x^3}{3} + \frac{x^2}{2} + 6x + C$

(c) $\int f(x)dx = F(x) + C$ where $F(x) = \begin{cases} \frac{x^3}{3} + \frac{x^2}{2} - 6x & \text{for } x \geq 2 \\ -\frac{x^3}{3} - \frac{x^2}{2} + 6x + \frac{44}{3} & \text{for } -3 \leq x < 2 \\ \frac{x^3}{3} + \frac{x^2}{2} - 6x - \frac{37}{3} & \text{for } x < -3 \end{cases}$

(d) $\int f(x)dx = F(x) + C$ where $F(x) = \begin{cases} x^2 + x & \text{for } x \geq 2 \\ 5x - 4 & \text{for } -3 \leq x < 2 \\ -x^2 - x - 13 & \text{for } x < -3 \end{cases}$

(e) $\int f(x)dx = F(x) + C$ where $F(x) = \begin{cases} \frac{\ln|2x+1|}{2} & \text{for } x \geq 2 \\ \frac{x}{5} + \frac{\ln 5}{2} - \frac{2}{5} & \text{for } -3 \leq x < 2 \\ -\frac{\ln|2x+1|}{2} + \ln 5 - 1 & \text{for } x < -3 \end{cases}$

□

Exercise 4. (Level 3)
Find

(a) $\int \frac{1}{\sin^2 x \cos^2 x}dx$ and

(b) $\int \frac{1}{\sqrt{ax+b}-\sqrt{ax+c}}dx$ provided that $b \neq c$.

Solution. (a)

$$\begin{aligned}\int \frac{1}{\sin^2 x\cos^2 x}dx &= \int \frac{\sin^2 x+\cos^2 x}{\sin^2 x\cos^2 x}dx \\ &= \int \frac{1}{\sin^2 x}+\frac{1}{\cos^2 x}dx \\ &= \tan x-\cot x+C\end{aligned}$$

(b)

$$\begin{aligned}\int \frac{1}{\sqrt{ax+b}-\sqrt{ax+c}}dx &= \int \frac{\sqrt{ax+b}+\sqrt{ax+c}}{(ax+b)-(ax+c)}dx \\ &= \frac{2}{3a(b-c)}\left[(ax+b)^{\frac{3}{2}}+(ax+c)^{\frac{3}{2}}\right]+C\end{aligned}$$

□

7.2 Integration by substitution

Exercise 5. (Level 2)
Evaluate the integrals.

(a) $\int \frac{x}{\sqrt{1+x^2}}dx$

(b) $\int \sqrt{x^3+1}x^2dx$

(c) $\int e^x\cos e^x dx$

(d) $\int \frac{\ln^3 x}{x}dx$

Solution. (a) $\int \frac{x}{\sqrt{1+x^2}}dx=\frac{1}{2}\int\frac{1}{\sqrt{1+x^2}}d(1+x^2)=\sqrt{1+x^2}+C$

(b) $\int \sqrt{x^3+1}x^2dx=\frac{1}{3}\int\sqrt{x^3+1}d(1+x^3)=\frac{2}{9}(x^3+1)^{\frac{3}{2}}+C$

(c) $\int e^x\cos e^x dx=\int\cos e^x de^x=\sin e^x+C$

(d) $\int \frac{\ln^3 x}{x}dx=\int\frac{\ln^3 x}{x}d\ln x=\frac{\ln^4 x}{4}+C$

□

Exercise 6. (Level 3)
Evaluate the following integrals.

(a) $\int 1+\tan^6 x dx$

(b) $\int \frac{x^{2n-1}}{x^n+1}dx$

(c) $\int \frac{1}{x+\sqrt{x^2+1}}dx$

(d) $\int \frac{\ln x}{x\sqrt{1+\ln x}}dx$

Solution. (a)

$$\begin{aligned}\int 1+\tan^6 x dx &= \int(\tan^4 x-\tan^2 x+1)(1+\tan^2 x)dx\\ &= \int(\tan^4 x-\tan^2 x+1)(\sec^2 x)dx\\ &= \int(\tan^4 x-\tan^2 x+1)d\tan x\\ &= \frac{\tan^5 x}{5}-\frac{\tan^3 x}{3}+\tan x+C\end{aligned}$$

(b)

$$\begin{aligned}\int\frac{x^{2n-1}}{x^n+1}dx &= \int\frac{x^{n-1}(x^n+1)-x^{n-1}}{x^n+1}dx\\ &= \int x^{n-1}-\frac{x^{n-1}}{x^n+1}dx\\ &= \frac{x^n}{n}-\frac{1}{n}\int\frac{1}{x^n+1}d(x^n+1)\\ &= \frac{x^n}{n}-\frac{\ln(x^n+1)}{n}+C.\end{aligned}$$

(c) Let $t=x+\sqrt{x^2+1}$, then $x=\frac{t^2-1}{2t}$ and $dx=\frac{t^2+1}{2t^2}dt$. We have

$$\begin{aligned}\int\frac{1}{x+\sqrt{x^2+1}}dx &= \int\frac{t^2+1}{2t^3}dt\\ &= \frac{1}{2}\int\frac{1}{t}+\frac{1}{t^3}dt\\ &= \frac{1}{2}\left(\ln|t|-\frac{1}{2t^2}\right)+C\\ &= \frac{1}{2}\left(\ln|x+\sqrt{x^2+1}|-\frac{1}{2(x+\sqrt{x^2+1})^2}\right)+C.\end{aligned}$$

(d) Let $t=1+\ln x$, then $\frac{\ln x}{x}dx=(t-1)dt$. We have

$$\int\frac{\ln x}{x\sqrt{1+\ln x}}dx=\int t^{-\frac{1}{2}}(t-1)dt=\frac{2}{3}(t-3)t^{\frac{1}{2}}+C=\frac{2}{3}(\ln x-2)(1+\ln x)^{\frac{1}{2}}+C.$$

□

Exercise 7. (Level 3/Level 4)
Using $\cos(A-B)=\cos A\cos B+\sin A\sin B$, find the following integrals.

(a) $\int\frac{dx}{\sin(x+a)\cos(x+b)}$ if $\cos(a-b)\neq 0$.

(b) $\int\frac{\cos^{n-1}\frac{x+a}{2}}{\sin^{n+1}\frac{x-a}{2}}dx$ if $\cos a\neq 0$.

Solution. (a)

$$\begin{aligned}\int \frac{dx}{\sin(x+a)\cos(x+b)} &= \frac{1}{\cos(a-b)}\int \frac{\cos(a-b)}{\sin(x+a)\cos(x+b)}dx \\ &= \frac{1}{\cos(a-b)}\int \frac{\cos[(x+a)-(x+b)]}{\sin(x+a)\cos(x+b)}dx \\ &= \frac{1}{\cos(a-b)}\int \frac{\cos(x+a)}{\sin(x+a)}+\frac{\sin(x+b)}{\cos(x+b)}dx \\ &= \frac{1}{\cos(a-b)}\int \frac{d\sin(x+a)}{\sin(x+a)}-\int \frac{d\cos(x+b)}{\cos(x+b)} \\ &= \frac{1}{\cos(a-b)}\ln\left|\frac{\sin(x+a)}{\cos(x+b)}\right|+C\end{aligned}$$

(b) Let $t = \frac{\cos\frac{x+a}{2}}{\sin\frac{x-a}{2}}$, then $dt = -\frac{\sin\frac{x+a}{2}\sin\frac{x-a}{2}+\cos\frac{x+a}{2}\cos\frac{x-a}{2}}{2\sin^2\frac{x+a}{2}}dx = -\frac{\cos a}{2\sin^2\frac{x+a}{2}}dx$. One obtains

$$\begin{aligned}\int \frac{\cos^{n-1}\frac{x+a}{2}}{\sin^{n+1}\frac{x-a}{2}}dx &= -\frac{1}{\cos a}\int 2t^{n-1}dt \\ &= -\frac{t^n}{n\cos a}+C \\ &= -\frac{1}{n\cos a}\frac{\cos^n\frac{x+a}{2}}{\sin^n\frac{x-a}{2}}+C.\end{aligned}$$

□

7.3 Integration by parts

Exercise 8. (Level 2)
Evaluate the following integrals:

(a) $\int x\sec^2 x dx$

(b) $\int \ln x dx$

(c) $\int \sec^2 x \ln\tan x dx$

Solution. (a)

$$\int x\sec^2 x dx = \int x d\tan x = x\tan x - \int \tan x dx = x\tan x - \ln|\sec x| + C$$

(b)

$$\int \ln x dx = x\ln x - \int x d\ln x = x\ln x - \int 1 dx = x\ln x - x + C$$

(c)

$$\int \sec^2 x\ln\tan x dx = \int \ln\tan x d\tan x = \tan x\ln\tan x - \tan x + C$$

□

Exercise 9. (Level 3/Level 4)
Evaluate the following integrals:

(a) $\int \sin^2 x dx$

(b) $\int \frac{x^5}{(1+x^3)^3} dx$

(c) $\int x^2 \ln \frac{1+x}{1-x} dx$

(d) $\int e^x \left(\frac{1+\sin x}{1+\cos x}\right) dx$

(e) $\int \sin \ln x dx$

Solution. (a)

$$\begin{aligned}
\int \sin^2 x dx &= -\int \sin x d\cos x \\
&= -\sin x \cos x + \int \cos^2 x dx \\
&= -\sin x \cos x + \int 1 - \sin^2 x dx \\
&= -\sin x \cos x + x - \int \sin^2 x dx
\end{aligned}$$

gives that

$$\int \sin^2 x dx = -\frac{\sin x \cos x}{2} + \frac{x}{2} + C.$$

(b)

$$\begin{aligned}
\int \frac{x^5}{(1+x^3)^3} dx &= \int \frac{-x^3}{6} d(1+x^3)^{-2} \\
&= -\frac{x^3}{6(1+x^3)^2} + \int \frac{x^2}{2(1+x^3)^2} dx \\
&= -\frac{x^3}{6(1+x^3)^2} + \int \frac{1}{6(1+x^3)^2} d(1+x^3) \\
&= -\frac{x^3}{6(1+x^3)^2} - \frac{1}{6(1+x^3)} + C
\end{aligned}$$

(c)

$$\begin{aligned}
\int x^2 \ln \frac{1+x}{1-x} dx &= \frac{1}{3} \int \ln \frac{1+x}{1-x} dx^3 \\
&= \frac{x^3}{3} \ln \frac{1+x}{1-x} - \frac{1}{3} \int x^3 d \ln \frac{1+x}{1-x} \\
&= \frac{x^3}{3} \ln \frac{1+x}{1-x} - \frac{2}{3} \int \frac{x^3}{1-x^2} dx \\
&= \frac{x^3}{3} \ln \frac{1+x}{1-x} - \frac{2}{3} \int \frac{x(x^2-1)+x}{1-x^2} dx \\
&= \frac{x^3}{3} \ln \frac{1+x}{1-x} + \frac{2}{3} \int x dx - \frac{2}{3} \int \frac{x}{1-x^2} dx \\
&= \frac{x^3}{3} \ln \frac{1+x}{1-x} + \frac{x^2}{3} + \frac{1}{3} \int \frac{1}{1-x^2} d(1-x^2) \\
&= \frac{x^3}{3} \ln \frac{1+x}{1-x} + \frac{x^2}{3} + \frac{1}{3} \ln(1-x^2) + C
\end{aligned}$$

(d)

$$\begin{aligned}
& \int e^x \left(\frac{1+\sin x}{1+\cos x} \right) dx \\
= & \int \left(\frac{1+\sin x}{1+\cos x} \right) de^x \\
= & \; e^x \left(\frac{1+\sin x}{1+\cos x} \right) - \int e^x d \left(\frac{1+\sin x}{1+\cos x} \right) \\
= & \; e^x \left(\frac{1+\sin x}{1+\cos x} \right) - \int e^x \left(\frac{1+\sin x+\cos x}{(1+\cos x)^2} \right) dx \\
= & \; e^x \left(\frac{1+\sin x}{1+\cos x} \right) - \int e^x \left(\frac{1}{1+\cos x} \right) dx - \int e^x \left(\frac{\sin x}{1+\cos x} \right) dx \\
= & \; e^x \left(\frac{1+\sin x}{1+\cos x} \right) - \int e^x \left(\frac{1}{1+\cos x} \right) dx - \int e^x d \left(\frac{1}{1+\cos x} \right) \\
= & \; e^x \left(\frac{1+\sin x}{1+\cos x} \right) - \int e^x \left(\frac{1}{1+\cos x} \right) dx - e^x \left(\frac{1}{1+\cos x} \right) + \int e^x \left(\frac{1}{1+\cos x} \right) dx \\
= & \; e^x \left(\frac{\sin x}{1+\cos x} \right) + C
\end{aligned}$$

(e) Note that

$$\begin{aligned} & \int \sin \ln x dx \\ = & \ x \sin \ln x - \int x d \sin \ln x \\ = & \ x \sin \ln x - \int \cos \ln x dx \\ = & \ x \sin \ln x - x \cos \ln x + \int x d \cos \ln x \\ = & \ x \sin \ln x - x \cos \ln x - \int x \sin \ln x dx, \end{aligned}$$

then, by rearrangement, we get

$$\int \sin \ln x dx = \frac{1}{2}(x \sin \ln x - x \cos \ln x) + C.$$

□

7.4 Trigonometric substitution

Exercise 10. (Level 2)
Evaluate the following integrals:

(a) $\int \frac{1}{x^2+9} dx$

(b) $\int \frac{1}{\sqrt{x^2+9}} dx$

(c) $\int \sqrt{x^2+9} dx$

(d) $\int \frac{1}{\sqrt{25-x^2}} dx$

(e) $\int \sqrt{25-x^2} dx$

(f) $\int \frac{1}{\sqrt{x^2-16}} dx$

(g) $\int \sqrt{x^2-16} dx$

Solution. (a) Let $x = 3\tan z$. Then $dx = 3\sec^2 z dz$, we get

$$\int \frac{1}{x^2+9} dx = \int \frac{1}{9\sec^2 z} \cdot 3\sec^2 z dz = 3z + C = 3\tan^{-1} \frac{x}{3} + C.$$

(b) Let $x = 3\tan z$. Then $dx = 3\sec^2 z dz$, we get

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2+9}}dx &= \int \frac{1}{3\sec z} \cdot 3\sec^2 z dz \\
&= \int \sec z dz \\
&= \ln|\tan z + \sec z| + C' \\
&= \ln|\frac{x}{3} + \frac{\sqrt{x^2+9}}{3}| + C' \\
&= \ln|x + \sqrt{x^2+9}| + C.
\end{aligned}$$

(c)

$$\begin{aligned}
\int \sqrt{x^2+9}dx &= x\sqrt{x^2+9} - \int x d\sqrt{x^2+9} \\
&= x\sqrt{x^2+9} - \int \frac{x^2}{\sqrt{x^2+9}}dx \\
&= x\sqrt{x^2+9} - \int \frac{x^2+9}{\sqrt{x^2+9}}dx + \int \frac{9}{\sqrt{x^2+9}}dx \\
&= x\sqrt{x^2+9} - \int \sqrt{x^2+9}dx + 9\ln|x + \sqrt{x^2+9}|
\end{aligned}$$

gives that

$$\int \sqrt{x^2+9}dx = \frac{x\sqrt{x^2+9}}{2} + \frac{9}{2}\ln|x + \sqrt{x^2+9}| + C$$

(d) Let $x = 5\sin z$. Then $dx = 5\cos z dz$, we get

$$\int \frac{1}{\sqrt{25-x^2}}dx = \int \frac{1}{5\cos z} \cdot 5\cos z dz = z + C = \sin^{-1}\frac{x}{5} + C.$$

(e)

$$\begin{aligned}
\int \sqrt{25-x^2}dx &= x\sqrt{25-x^2} - \int x d\sqrt{25-x^2} \\
&= x\sqrt{25-x^2} + \int \frac{x^2}{\sqrt{25-x^2}}dx \\
&= x\sqrt{25-x^2} + \int \frac{x^2-25}{\sqrt{25-x^2}}dx + \int \frac{25}{\sqrt{25-x^2}}dx \\
&= x\sqrt{25-x^2} - \int \sqrt{25-x^2}dx + \int \frac{25}{\sqrt{25-x^2}}dx
\end{aligned}$$

gives that

$$\int \sqrt{25-x^2}dx = \frac{x\sqrt{25-x^2}}{2} + \frac{25}{2}\sin^{-1}\frac{x}{5} + C.$$

(f) Let $x = 4\sec z$. Then $dx = 4\sec z \tan z dz$, we get

$$\begin{aligned}\int \frac{1}{\sqrt{x^2-16}}dx &= \int \frac{1}{4\tan z}\cdot 4\sec z \tan z dz \\ &= \int \sec z dz \\ &= \ln|\sec z + \tan z| + C' \\ &= \ln|\frac{x}{4} + \frac{\sqrt{x^2-16}}{4}| + C' \\ &= \ln|x + \sqrt{x^2-16}| + C.\end{aligned}$$

(g)

$$\begin{aligned}\int \sqrt{x^2-16}dx &= x\sqrt{x^2-16} - \int x d\sqrt{x^2-16} \\ &= x\sqrt{x^2-16} - \int \frac{x^2}{\sqrt{x^2-16}}dx \\ &= x\sqrt{x^2-16} - \int \frac{x^2-16}{\sqrt{x^2-16}}dx - 16\int \frac{1}{\sqrt{x^2-16}}dx \\ &= x\sqrt{x^2-16} - \int \sqrt{x^2-16}dx - 16\int \frac{1}{\sqrt{x^2-16}}dx\end{aligned}$$

gives that

$$\int \sqrt{x^2-16}dx = \frac{x\sqrt{x^2-16}}{2} - 8\ln|x + \sqrt{x^2-16}| + C.$$

□

Exercise 11. (Level 3)
Evaluate the following integrals:

(a) $\int \frac{\sqrt{x^2-a^2}}{x^4}dx$

(b) $\int \frac{1}{x^2+x+1}dx$

(c) $\int \frac{2\sin x}{3\sin^2 x+4\cos^2 x}dx$

(d) $\int \frac{1}{x^2-2x\cos\alpha+1}dx$ $(0 < \alpha < \pi)$

(e) $\int \frac{(\sin^{-1} x)^2}{\sqrt{1-x^2}}dx$

(f) $\int \frac{1}{(1+x^2)\sqrt{1-x^2}}dx$

Solution. (a) Let $x = a\sec z$, then $dx = a\sec x\tan z dz$

$$\begin{aligned}
\int \frac{\sqrt{x^2-a^2}}{x^4}dx &= \int \frac{a\tan z}{a^4\sec^4 z} a\sec x\tan z dz \\
&= \frac{1}{a^2}\int \sin^2 z\cos z dz \\
&= \frac{1}{a^2}\int \sin^2 z d\sin z \\
&= \frac{1}{a^2}\frac{\sin^3 z}{3} + C \\
&= \frac{(x^2-a^2)^{\frac{3}{2}}}{3a^2x^3} + C
\end{aligned}$$

(b)

$$\int \frac{1}{x^2+x+1}dx = \int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}dx = \frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + C.$$

(c)

$$\int \frac{2\sin x}{3\sin^2 x + 4\cos^2 x}dx = -2\int \frac{d\cos x}{3+\cos^2 x} = -\frac{2}{\sqrt{3}}\tan^{-1}\frac{\cos x}{\sqrt{3}} + C.$$

(d)

$$\begin{aligned}
\int \frac{1}{x^2-2x\cos\alpha+1}dx &= \int \frac{1}{x^2-2x\cos\alpha+\cos^2\alpha+\sin^2\alpha}dx \\
&= \int \frac{1}{(x-\cos\alpha)^2+\sin^2\alpha}dx \\
&= \frac{1}{\sin\alpha}\tan^{-1}\frac{x-\cos\alpha}{\sin\alpha} + C
\end{aligned}$$

(e) Let $x = \sin z$, then $dx = \cos z dz$. We get

$$\int \frac{(\sin^{-1}x)^2}{\sqrt{1-x^2}}dx = \int \frac{z^2\cos z dz}{\sqrt{1-\sin^2 z}} = \int z^2 dz = \frac{z^3}{3} + C = \frac{(\sin^{-1}x)^3}{3} + C.$$

(f) Let $x = \sin z$, then $dx = \cos z dz$. We get

$$\begin{aligned}
\int \frac{1}{(1+x^2)\sqrt{1-x^2}}dx &= \int \frac{1}{(1+\sin^2 z)}dz \\
&= \int \frac{\csc^2 z}{(1+\csc^2 z)}dz \\
&= -\int \frac{1}{(2+\cot^2 z)}d\cot z \\
&= -\frac{1}{\sqrt{2}}\tan^{-1}\frac{\cot z}{\sqrt{2}} + C \\
&= -\frac{1}{\sqrt{2}}\tan^{-1}\sqrt{\frac{1-x^2}{2x^2}} + C.
\end{aligned}$$

□

7.5 Integration of rational functions

Exercise 12. (Level 2)
Find $\int \frac{dx}{x^2-100}$

Solution. Put $\frac{1}{x^2-100} = \frac{A}{x-10} + \frac{B}{x+10}$. Then we have $1 = A(x+10) + B(x-10)$, hence $A = \frac{1}{20}$ and $B = -\frac{1}{20}$. Thus,

$$\begin{aligned}\int \frac{dx}{x^2-100} &= \int \frac{1}{20(x-10)} - \frac{1}{20(x+10)} dx \\ &= \frac{1}{20}\ln|x-10| - \frac{1}{20}\ln|x+10| + C \\ &= \frac{1}{20}\ln\left|\frac{x-10}{x+10}\right| + C.\end{aligned}$$

□

Exercise 13. (Level 3/Level 4)

(a) (i) Show that $\frac{1}{x^3+1} = \frac{1}{3(1+x)} + \frac{2-x}{3(1-x+x^2)}$.

(ii) Find $\int \frac{1}{x^3+1} dx$.

(b) (i) Show that $\frac{1}{x^4+1} = \frac{x+\sqrt{2}}{2\sqrt{2}(x^2+\sqrt{2}x+1)} - \frac{x-\sqrt{2}}{2\sqrt{2}(x^2-\sqrt{2}x+1)}$.

(ii) Find $\int \frac{1}{x^4+1} dx$.

Solution. [(Level 3/Level 4)]

(a) (i) The solution is omitted since it is straightforward.

(ii) Noting that

$$\begin{aligned}\int \frac{2-x}{3(1-x+x^2)} dx &= \int \frac{-1}{6}\frac{2x-1}{1-x+x^2} + \frac{1}{2}\frac{1}{1-x+x^2} dx \\ &= -\frac{1}{6}\int \frac{d(1-x+x^2)}{1-x+x^2} + \frac{1}{2}\int \frac{dx}{(x-\frac{1}{2})^2+\frac{3}{2}} \\ &= -\frac{1}{6}\ln(1-x+x^2) + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}} + C,\end{aligned}$$

and using (a)(i), we get

$$\begin{aligned}\int \frac{1}{x^3+1} dx &= \int \frac{1}{3(1+x)} + \frac{2-x}{3(1-x+x^2)} dx \\ &= \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln(1-x+x^2) + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}} + C.\end{aligned}$$

(b) (i) The solution is omitted since it is straightforward.

(ii) Noting that

$$\begin{aligned}
&\int \frac{x+\sqrt{2}}{2\sqrt{2}(x^2+\sqrt{2}x+1)}dx \\
=\; & \frac{1}{4\sqrt{2}}\int \frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1} + \frac{\sqrt{2}}{x^2+\sqrt{2}x+1}dx \\
=\; & \frac{1}{4\sqrt{2}}\int \frac{d(x^2+\sqrt{2}x+1)}{x^2+\sqrt{2}x+1} + \frac{1}{4}\int \frac{1}{(x+\frac{1}{\sqrt{2}})^2+\frac{1}{2}}dx \\
=\; & \frac{1}{4\sqrt{2}}\ln(x^2+\sqrt{2}x+1) + \frac{1}{2\sqrt{2}}\tan^{-1}(2x+1) + C_1
\end{aligned}$$

and

$$\begin{aligned}
&\int \frac{-x+\sqrt{2}}{2\sqrt{2}(x^2-\sqrt{2}x+1)}dx \\
=\; & \int -\frac{y+\sqrt{2}}{2\sqrt{2}(y^2+\sqrt{2}y+1)}dy \\
=\; & -\frac{1}{4\sqrt{2}}\ln(y^2+\sqrt{2}y+1) - \frac{1}{2\sqrt{2}}\tan^{-1}(2y+1) + C_2 \\
=\; & -\frac{1}{4\sqrt{2}}\ln(x^2-\sqrt{2}x+1) + \frac{1}{2\sqrt{2}}\tan^{-1}(2x-1) + C_2,
\end{aligned}$$

one obtains

$$\begin{aligned}
&\int \frac{1}{x^4+1}dx \\
=\; & \int \frac{x+\sqrt{2}}{2\sqrt{2}(x^2+\sqrt{2}x+1)} - \frac{x-\sqrt{2}}{2\sqrt{2}(x^2-\sqrt{2}x+1)}dx \\
=\; & \frac{1}{4\sqrt{2}}\ln(x^2+\sqrt{2}x+1) + \frac{1}{2\sqrt{2}}\tan^{-1}(2x+1) \\
& -\frac{1}{4\sqrt{2}}\ln(x^2-\sqrt{2}x+1) + \frac{1}{2\sqrt{2}}\tan^{-1}(2x-1) + C \\
=\; & \frac{1}{4\sqrt{2}}\ln\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1} + \frac{1}{2\sqrt{2}}\tan^{-1}(2x+1) + \frac{1}{2\sqrt{2}}\tan^{-1}(2x-1) + C.
\end{aligned}$$

□

7.6 Reduction formulae

Exercise 14. (Level 3)
Let $I_{m,n} = \int \sin^m x \cos^n x dx$.

(a) Derive the following reduction formulae for $m+n \neq 0$

(i)

$$I_{m,n} = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}$$

(ii)

$$I_{m,n} = -\frac{\sin^{m+1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m,n-2}$$

(b) Find $\int \sin^4 x \cos^4 x dx$.

Solution. (a) (i) Using integration by parts,

$$\begin{aligned} I_{m,n} &= \int \sin^m x \cos^n x dx \\ &= \frac{1}{m+1} \int \cos^{n-1} x d \sin^{m+1} x \\ &= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \sin^{m+2} x \cos^{n-2} x dx \\ &= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \sin^m x (1-\cos^2 x) \cos^{n-2} x dx \\ &= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2} - \frac{n-1}{m+1} I_{m,n} \end{aligned}$$

gives the result by a simple rearrangement.

(ii) Using integration by parts,

$$\begin{aligned} I_{m,n} &= \int \sin^m x \cos^n x dx \\ &= -\frac{1}{n+1} \int \sin^{m-1} x d \cos^{n+1} x \\ &= -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1} \int \sin^{m-2} x \cos^{n+2} x dx \\ &= -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1} \int \sin^{m-2} x \cos^n x (1-\sin^2 x) dx \\ &= -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1} I_{m-2,n} - \frac{m-1}{n+1} I_{m,n} \end{aligned}$$

gives the result by a simple rearrangement.

(b) Using the previous results, one obtains,

$$\begin{aligned} & I_{4,4} \\ = & \frac{\sin^5 x \cos^3 x}{8} + \frac{3}{8} I_{4,2} \\ = & \frac{\sin^5 x \cos^3 x}{8} + \frac{3}{8}\left[-\frac{\sin^3 x \cos^3 x}{8} + \frac{3}{6} I_{2,2}\right] \\ = & \frac{\sin^5 x \cos^3 x}{8} - \frac{\sin^3 x \cos^3 x}{16} + \frac{3}{16}\left[\frac{\sin^3 x \cos x}{4} + \frac{1}{4} I_{2,0}\right] \\ = & \frac{\sin^5 x \cos^3 x}{8} - \frac{\sin^3 x \cos^3 x}{16} + \frac{3 \sin^3 x \cos x}{64} + \frac{3}{64}\left[-\frac{\sin x \cos x}{2} + \frac{1}{2} I_{0,0}\right] \\ = & \frac{\sin^5 x \cos^3 x}{8} - \frac{\sin^3 x \cos^3 x}{16} + \frac{3 \sin^3 x \cos x}{64} - \frac{3 \sin x \cos x}{128} + \frac{3}{128} x + C. \end{aligned}$$

□

8 Definite Integration

8.1 Mean value theorem for integrals

Exercise 1. (Level 2)

Given that $\int_0^1 \frac{1}{x^2+1}dx = \frac{\pi}{4}$ and $\int_0^1 \sin \pi x dx = \frac{2}{\pi}$. Show that there exist two points a and b in the closed interval $[0,1]$ such that

$$\int_0^1 \frac{\sin \pi x}{x^2+1} = \frac{2}{\pi(a^2+1)} = \frac{\pi}{4}\sin \pi b.$$

Solution. Both functions $\frac{1}{x^2+1}$ and $\sin \pi x$ are continuous and positive for over $(0,1)$. Applying the mean value theorem twice, we get that
there exist two points a and b in the closed interval $[0,1]$ such that

$$\int_0^1 \frac{\sin \pi x}{x^2+1} = \frac{1}{a^2+1}\int_0^1 \sin \pi x dx = \frac{2}{\pi(a^2+1)},$$

and

$$\int_0^1 \frac{\sin \pi x}{x^2+1} = \sin \pi b \int_0^1 \frac{1}{x^2+1}dx = \frac{\pi}{4}\sin \pi b.$$

□

8.2 Fundamental theorem of Calculus

Exercise 2. (Level 2)

Evaluate the following integrals:

(a) $\int_{-4}^{20} y+5-\frac{y^2}{16}dy$

(b) $\int_1^2 2y+3-\frac{1}{4(2x+3)}dy$

(c) $\int_0^2 |1-y|dy$

(d) $\int_0^{\frac{\pi}{2}} \sec^3 \frac{y}{2}\tan\frac{y}{2}dy$

(e) $\int_{-2017}^{2017} y^{2017}(y^{7102}+7201)y^{\frac{1}{2017}}\sin^{2017} y \ln|y^{7102}+2017|dy$

Solution. (a)

$$\int_{-4}^{20} y+5-\frac{y^2}{16}dy = \left[\frac{y^2}{2}+5y-\frac{y^3}{48}\right]_{-4}^{20} = 144$$

(b)

$$\int_1^2 2y+3-\frac{1}{4(2x+3)}dy = \left[y^2+3y+\frac{1}{8}\ln|2x+3|\right]_1^2 = 6+\frac{1}{8}\ln|\frac{7}{5}|$$

(c)

$$\int_0^2 |1-y|dy = \int_0^1 1-ydy + \int_1^2 y-1dy = \left[y - \frac{1}{2}y^2\right]_0^1 + \left[\frac{1}{2}y^2 - y\right]_1^2 = 1$$

(d)

$$\int_0^{\frac{\pi}{2}} \sec^3\frac{y}{2}\tan\frac{y}{2}dy = 2\int_0^{\frac{\pi}{2}} \sec^2\frac{y}{2}d\sec\frac{y}{2} = \frac{2}{3}\left[\sec^3\frac{y}{2}\right]_0^{\frac{\pi}{2}} = \frac{2\sqrt{8}-2}{3}$$

(e) Since the integrand is an odd function, the integral is 0.

□

Exercise 3. (Level 3)
Evaluate the following integrals:

(a) $\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{\sin^{-1}x\sqrt{1-x^2}}dx$

(b) $\int_0^{\ln 2} \sqrt{e^x - 1}dx$

(c) $\int_1^2 x^2 \ln x dx$

(d) $\int_1^2 x^2 \ln^2 x dx$

Solution. (a)

$$\begin{aligned}\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{\sin^{-1}x\sqrt{1-x^2}}dx &= \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{d\sin^{-1}x}{\sin^{-1}x}dx \\ &= \left[\ln\sin^{-1}x\right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \\ &= \ln 2\end{aligned}$$

(b) Let $\sec^2 y = e^x$, then $dx = 2\tan ydy$. When $x = 0$ or $x = \ln 2$, $y = 0$ or $y = \frac{\pi}{4}$ accordingly. We have that

$$\begin{aligned}\int_0^{\ln 2} \sqrt{e^x-1}dx &= \int_0^{\frac{\pi}{4}} \sqrt{\tan^2 y}\cdot 2\tan ydy \\ &= 2\int_0^{\frac{\pi}{4}} \tan^2 ydy \\ &= 2\int_0^{\frac{\pi}{4}} \sec^2 y - 1dy \\ &= 2\left[\tan y - y\right]_0^{\frac{\pi}{4}} \\ &= 2 - \frac{\pi}{2}\end{aligned}$$

(c)

$$\begin{aligned}\int_1^2 x^2 \ln x dx &= \int_1^2 \ln x d\frac{x^3}{3} \\ &= \left[\frac{x^3}{3}\ln x\right]_1^2 - \int_1^2 \frac{x^2}{3}dx \\ &= \frac{8}{3}\ln 2 - \left[\frac{x^3}{9}\right]_1^2 \\ &= \frac{8}{3}\ln 2 - \frac{7}{9}\end{aligned}$$

(d)

$$\begin{aligned}\int_1^2 x^2 \ln^2 x dx &= \int_1^2 \ln^2 x d\frac{x^3}{3} \\ &= \left[\frac{x^3}{3}\ln^2 x\right]_1^2 - \int_1^2 \frac{x^2}{3}d\ln^2 x \\ &= \frac{8}{3}\ln^2 2 - \int_1^2 \frac{2x^2}{3}\ln x dx \\ &= \frac{8}{3}\ln^2 2 - \frac{2}{3}\left(\frac{8}{3}\ln 2 - \frac{7}{9}\right) \\ &= \frac{8}{3}\ln^2 2 - \frac{16}{9}\ln 2 + \frac{14}{27}\end{aligned}$$

□

Exercise 4. (Level 3/Level 4)
Using some suitable substitution, show that

(a) $\int_0^{\frac{\pi}{2}} \frac{\sin^m x}{\sin^m x+\cos^m x}dx = \frac{\pi}{4}$;

(b) $\int_0^{\frac{\pi}{2}} \ln \frac{1+\sin x}{1+\cos x}dx = 0$;

(c) $\int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^\lambda x}dx = \frac{\pi}{4}$;

(d) $\int_0^1 \frac{1}{x+\sqrt{1-x^2}}dx = \frac{\pi}{4}$.

Solution. (a) By means of the substitution $x = \frac{\pi}{2} - y$,

$$\int_0^{\frac{\pi}{2}} \frac{\sin^m x}{\sin^m x + \cos^m x}dx = \int_0^{\frac{\pi}{2}} \frac{\cos^m x}{\sin^m x + \cos^m x}dx.$$

Thus,

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \frac{\sin^m x}{\sin^m x + \cos^m x} dx \\
= \;& \frac{1}{2}\left(\int_0^{\frac{\pi}{2}} \frac{\sin^m x}{\sin^m x + \cos^m x} dx + \int_0^{\frac{\pi}{2}} \frac{\cos^m x}{\sin^m x + \cos^m x} dx\right) \\
= \;& \frac{1}{2}\int_0^{\frac{\pi}{2}} 1 dx \\
= \;& \frac{\pi}{4}.
\end{aligned}$$

(b) Making use of the substitution $x = \frac{\pi}{2} - y$,

$$\int_0^{\frac{\pi}{2}} \ln(1+\sin x) dx = \int_0^{\frac{\pi}{2}} \ln(1+\cos x) dx.$$

We then have

$$\int_0^{\frac{\pi}{2}} \ln\frac{1+\sin x}{1+\cos x} dx = \int_0^{\frac{\pi}{2}} \ln(1+\sin x) dx - \int_0^{\frac{\pi}{2}} \ln(1+\cos x) dx = 0.$$

(c) By substitution $x = \frac{\pi}{2} - y$, it suggests us that

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^\lambda x} dx = \int_0^{\frac{\pi}{2}} \frac{1}{1+\cot^\lambda x} dx = \int_0^{\frac{\pi}{2}} \frac{\tan^\lambda x}{1+\tan^\lambda x} dx.$$

We hence get

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^\lambda x} dx &= \frac{1}{2}\left(\int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^\lambda x} dx + \int_0^{\frac{\pi}{2}} \frac{\tan^\lambda x}{1+\tan^\lambda x} dx\right) \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} 1 dx \\
&= \frac{\pi}{4}.
\end{aligned}$$

(d) Let $x = \sin z$, then $dx = \cos z dz$ and when $x = 0$, $z = 0$; when $x = 1$, $z = \frac{\pi}{2}$. One arrives that

$$\int_0^1 \frac{1}{x + \sqrt{1-x^2}} dx = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx = \frac{\pi}{4},$$

the last equality is justified by (a).

□

Exercise 5. (Level 2)
Find the derivative of the functions below.

(a) $\int_a^{2x} \sin t \ln(1+t) dt$

(b) $\int_b^{2x} \sin t \ln(1+x) dt$

(c) $\int_{2x}^{c} \sin t \ln(1+t)dt$

(d) $\int_{2x}^{2x} \sin t \ln(1+x)dt$

Solution. (a) The derivative is $2\sin(2x)\ln(1+2x)$.

(b) The derivative is $2\ln(1+x)\sin(2x) + \frac{\int_b^{2x} \sin t dt}{1+x} = 2\ln(1+x)\sin(2x) + \frac{\cos b - \cos(2x)dt}{1+x}$.

(c) The derivative is $-2\sin(2x)\ln(1+2x)$.

(d) The derivative is 0.

□

Exercise 6. (Level 2/Level 3)
Define for $-\frac{\pi}{2} < x < \frac{\pi}{2}$,

$$F(x) = \int_{\tan x}^{\sec x} (\sqrt{2})^{t^2} dt.$$

Solve $F'(x) = 0$.

Solution. Write $F(x) = \int_{\tan x}^{\sec x} (\sqrt{2})^{t^2} dt = \int_0^{\sec x} (\sqrt{2})^{t^2} dt - \int_0^{\tan x} (\sqrt{2})^{t^2} dt$, then we have

$$F'(x) = (\sqrt{2})^{\sec^2 x} \sec x \tan x - (\sqrt{2})^{\tan^2 x} \sec^2 x.$$

Then one has

$$\begin{aligned} F'(x) &= 0 \\ (\sqrt{2})^{\sec^2 x} \sec x \tan x - (\sqrt{2})^{\tan^2 x} \sec^2 x &= 0 \\ \frac{\tan x}{\sec x} &= (\sqrt{2})^{\tan^2 x - \sec^2 x} \\ \sin x &= \frac{1}{\sqrt{2}} \\ x &= \frac{\pi}{4}. \end{aligned}$$

□

Exercise 7. (Level 3)
Evaluate the following limits

(a) $\lim_{x\to 0} \left(\frac{1}{x^4} \int_0^{x^2} \frac{\sin^2 t}{t^2} \ln(1+t)dt \right)$

(b) $\lim_{x\to 0} \left(\frac{1}{x^3} \int_0^{x^2} \frac{\sin^2 t}{t^2} \ln(1+x)dt \right)$

(c) $\lim_{x\to 0} \left(\frac{1}{x\int_0^x \sin t^2 dt} - \frac{3}{x^4} \right)$

Solution. (a) Noting

$$\lim_{x\to 0}\frac{\sin^2 x^2}{x^4}=1$$

and

$$\lim_{x\to 0}\frac{\ln(1+x^2)}{x^2}=\lim_{x\to 0}\frac{\frac{2x}{1+x^2}}{2x}=1,$$

one gets

$$\begin{aligned}\lim_{x\to 0}\left(\frac{1}{x^4}\int_0^{x^2}\frac{\sin^2 t}{t^2}\ln(1+t)dt\right) &= \lim_{x\to 0}\frac{(\frac{\sin^2 x^2}{x^4}\ln(1+x^2))(2x)}{4x^3}\\ &= \frac{1}{2}\lim_{x\to 0}\frac{\sin^2 x^2}{x^4}\cdot\lim_{x\to 0}\frac{\ln(1+x^2)}{x^2}\\ &= \frac{1}{2}.\end{aligned}$$

(b) Noting

$$\lim_{x\to 0}\left(\frac{1}{x^2}\int_0^{x^2}\frac{\sin^2 t}{t^2}dt\right)=\lim_{x\to 0}\frac{2x(\frac{\sin^2 x^2}{x^4})}{2x}=1$$

and

$$\lim_{x\to 0}\frac{\ln(1+x)}{x}=\lim_{x\to 0}\frac{1}{1+x}=1,$$

one gets

$$\begin{aligned}\lim_{x\to 0}\left(\frac{1}{x^3}\int_0^{x^2}\frac{\sin^2 t}{t^2}\ln(1+x)dt\right) &= \lim_{x\to 0}\left(\frac{1}{x^2}\int_0^{x^2}\frac{\sin^2 t}{t^2}dt\right)\cdot\lim_{x\to 0}\frac{\ln(1+x)}{x}\\ &= 1.\end{aligned}$$

(c) Noting

$$\lim_{x\to 0}\frac{x^3}{\int_0^x \sin t^2 dt}=\lim_{x\to 0}\left(\frac{3x^2}{\sin x^2}\right)=3$$

and

$$\begin{aligned}\lim_{x\to 0}\frac{x^3-3\int_0^x \sin t^2 dt}{x^7} &= \lim_{x\to 0}\frac{3x^2-3\sin x^2}{7x^6}\\ &= \lim_{x\to 0}\frac{6x-6x\cos x^2}{42x^5}\\ &= \lim_{x\to 0}\frac{1-\cos x^2}{7x^4}\\ &= \lim_{x\to 0}\frac{2x\sin x^2}{28x^3}\\ &= \frac{1}{14},\end{aligned}$$

it gives

$$\begin{aligned}\lim_{x\to 0}\left(\frac{1}{x\int_0^x \sin t^2 dt}-\frac{3}{x^4}\right) &= \lim_{x\to 0}\frac{x^3-3\int_0^x \sin t^2 dt}{x^4\int_0^x \sin t^2 dt}\\ &= \lim_{x\to 0}\frac{x^3-3\int_0^x \sin t^2 dt}{x^7}\cdot \lim_{x\to 0}\frac{x^3}{\int_0^x \sin t^2 dt}\\ &= \frac{3}{14}.\end{aligned}$$

□

Exercise 8. (Level 3)

Let f is a continuous function on $\mathbf{R}$. Suppose $\lim\limits_{x\to+\infty}(\int_0^x f(t)dt+f(x))$ exists and that $\lim\limits_{x\to+\infty}(e^x\int_0^x f(t)dt)=\infty$. Show that $\lim\limits_{x\to+\infty} f(x)=0$.

Solution. By L'Hopital's rule and the fundamental theorem of calculus, then we get

$$\begin{aligned}\lim_{x\to+\infty}\int_0^x f(t)dt &= \lim_{x\to+\infty}\frac{e^x\int_0^x f(t)dt}{e^x}\\ &= \lim_{x\to+\infty}\frac{e^x(\int_0^x f(t)dt+f(x))}{e^x}\\ &= \lim_{x\to+\infty}(\int_0^x f(t)dt+f(x)).\end{aligned}$$

Hence $\lim\limits_{x\to+\infty}\int_0^x f(t)dt$ exists and $\lim\limits_{x\to+\infty} f(x)=0$.

□

Exercise 9. (Level 3/Level 4)

(a) For $x\in(-1,1)$, by considering $\int_0^x \frac{1}{1-t}dt$, show that

$$-\ln(1-x)=\sum_{n=1}^{\infty}\frac{x^n}{n}.$$

(b) Can you write down a function whose Taylor's series is

$$\sum_{n=1}^{\infty}\frac{x^n}{n^2}?$$

Solution. (a) Making use of geometric series, for $x\in(-1,1)$,

$$\int_0^x \frac{1}{1-t}dt=\int_0^x\sum_{n=0}^{\infty}t^n dt=\sum_{n=0}^{\infty}\int_0^x t^n dt=\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}=\sum_{n=1}^{\infty}\frac{x^n}{n}.$$

On the other hand, we have $\int_0^x \frac{1}{1-t}dt = -\ln(1-x)$. Hence

$$-\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}.$$

(b) A direction calculation yields that

$$\int_0^x \frac{-\ln(1-t)}{t}dt = \int_0^x \sum_{n=1}^{\infty} \frac{t^{n-1}}{n}dt = \sum_{n=1}^{\infty} \int_0^x \frac{t^{n-1}}{n}dt = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

for $x \in (-1, 1)$.

□

Exercise 10. (Level 5)

(a) For $0 < x < \frac{\pi}{2}$, show that $\ln \sec x < \frac{1}{2} \sin x \tan x$.

(b) Define $g(x) = \int_0^x (1 + \sec t)dt$ and $\rho(x) = \frac{8 \ln \sec x}{[g(x)]^2}$ for $x \in (0, \frac{\pi}{2})$.

- (i) Show that $g(x) > 0$ for $x \in (0, \frac{\pi}{2})$, and find $g(x)$.
- (ii) Show that $\rho(x)$ is decreasing.
  (Hint: Consider $[g(x)]^3 \rho'(x) \cot x$.)
- (iii) Find $\lim_{x \to 0^+} \rho(x)$.
- (iv) Find $\lim_{x \to \frac{\pi}{2}^-} \rho(x)$.

(c) Define $f(x) = \int_0^x (1 + \sec t) \ln \sec t dt$ and $\sigma(x) = \frac{f(x)}{g(x) \ln \sec x}$ for $x \in (0, \frac{\pi}{2})$.

- (i) Find $\lim_{x \to 0^+} \sigma(x)$.
- (ii) Find $\lim_{x \to \frac{\pi}{2}^-} \sigma(x)$.
- (iii) It is given the following fact: Let $u, v : [0, c) \to \mathbf{R}$ be two continuous functions and continuously differentiable on $(0, c)$, in which satisfied that u, v and $\frac{u'}{v'}$ are positive and increasing. Furthermore if $u(0) = v(0) = 0$, then $\frac{u}{v}$ is increasing.
  - A. Show that $1 - \csc x \cot x \ln \sec x$ is increasing.
  - B. Show that $\frac{(1+\sec x) \ln \sec x}{\tan x \int_0^x 1+\sec t dt}$ is increasing.
  - C. Find $\lim_{x \to 0^+} \frac{(1+\sec x) \ln \sec x}{\tan x \int_0^x 1+\sec t dt}$.
  - D. Show that $\sigma(x)$ is increasing.

Solution. (a) Let $h(x) = \ln \sec x - \frac{1}{2} \sin x \tan x$. Differentiation gives $h'(x) = \tan x \left(1 - \frac{\cos x + \sec x}{2}\right) = -\tan x \frac{(\cos x - 1)^2}{\cos x} < 0$ for $0 < x < \frac{\pi}{2}$. Hence $h(x) < h(0)$, that is, $\ln \sec x < \frac{1}{2} \sin x \tan x$.

(b) (i) Since the integrand $1 + \sec t > 0$ for $t \in (0, \frac{\pi}{2})$, $g(x) > 0$ for $x \in (0, \frac{\pi}{2})$. And $g(x) = \int_0^x 1 + \sec t dt = x + \ln(\sec x + \tan x)$.

(ii) Note that, by the fundamental theorem of calculus, $g'(x) = 1 + \sec x$. Then

$$\begin{aligned} \rho'(x) &= 8\frac{[g(x)]^2 \tan x - 2g'(x)g(x)\ln\sec x}{[g(x)]^4} \\ &= 8\frac{\tan x}{[g(x)]^3}(g(x) - 2(\csc x + \cot x)\ln\sec x) \end{aligned}$$

which gives

$$[g(x)]^3\rho'(x)\cot x = 8(g(x) - 2(\csc x + \cot x)\ln\sec x).$$

Note that $\lim_{x\to 0^+}\frac{\ln\sec x}{\sin x} = \lim_{x\to 0^+}\frac{\tan x}{\cos x} = 0$, and hence

$$\lim_{x\to 0^+}([g(x)]^3\rho'(x)\cot x) = \lim_{x\to 0^+} 8(g(x) - 2(\csc x + \cot x)\ln\sec x) = 0.$$

Consider $[g(x)]^3\rho'(x)\cot x$. Then, by (a),

$$\begin{aligned} &\frac{d[g(x)]^3\rho'(x)\cot x}{dx} \\ = \;& 8(1 + \sec x - 2(\csc x + \cot x)\tan x + 2\csc x(\csc x + \cot x)\ln\sec x) \\ < \;& 8(1 + \sec x - 2(\csc x + \cot x)\tan x + 2\csc x(\csc x + \cot x)\frac{1}{2}\sin x\tan x) \\ = \;& 0. \end{aligned}$$

One arrives that for $x \in (0, \frac{\pi}{2})$

$$[g(x)]^3\rho'(x)\cot x < \lim_{x\to 0^+}([g(x)]^3\rho'(x)\cot x) = 0$$

which implies that $\rho'(x) < 0$ provided that $g(x)$ and $\cot x$ both are positive on the interval. Thus $\rho(x)$ is decreasing.

(iii) We first compute $\lim_{x\to 0^+}\frac{\sin x}{g(x)} = \lim_{x\to 0^+}\frac{\cos x}{1+\sec x} = \frac{1}{2}$. Then,

$$\begin{aligned} \lim_{x\to 0^+}\rho(x) &= \lim_{x\to 0^+}\frac{8\ln\sec x}{[g(x)]^2} \\ &= \lim_{x\to 0^+}\frac{4\tan x}{g(x)(1+\sec x)} \\ &= \lim_{x\to 0^+}\left(\frac{4}{1+\cos x}\frac{\sin x}{g(x)}\right) \\ &= 1. \end{aligned}$$

(iv) Using $\lim_{x\to\frac{\pi}{2}^-} g(x) = \lim_{x\to\frac{\pi}{2}^-}(x + \ln(\sec x + \tan x)) = \infty$ we get

$$\lim_{x\to\frac{\pi}{2}^-}\rho(x) = \lim_{x\to\frac{\pi}{2}^-}\frac{8\ln\sec x}{[g(x)]^2} = \lim_{x\to\frac{\pi}{2}^-}\frac{4\sin x}{g(x)(1+\cos x)} = 0.$$

(c) Define $\sigma(x) = \frac{\int_0^x (1+\sec t)\ln\sec t\,dt}{g(x)\ln\sec x}$ for $x \in (0, \frac{\pi}{2})$.

(i) Note that, by the fundamental theorem of calculus,

$$\begin{aligned}
\lim_{x\to 0^+}\frac{\int_0^x(1+\sec t)\ln\sec t\,dt}{[g(x)]^3} &= \lim_{x\to 0^+}\frac{\ln\sec x}{3[g(x)]^2}\\
&= \lim_{x\to 0^+}\frac{\tan x}{6g(x)(1+\sec x)}\\
&= \lim_{x\to 0^+}\frac{\sin x}{6g(x)(1+\cos x)}\\
&= \lim_{x\to 0^+}\frac{\cos x}{6((1+\sec x)(1+\cos x)-g(x)\sin x))}\\
&= \frac{1}{24},
\end{aligned}$$

and

$$\sigma(x)=\frac{8\int_0^x(1+\sec t)\ln\sec t\,dt}{\rho(x)[g(x)]^3}$$

then we get

$$\begin{aligned}
\lim_{x\to 0^+}\sigma(x) &= \lim_{x\to 0^+}\frac{8\int_0^x(1+\sec t)\ln\sec t\,dt}{\rho(x)[g(x)]^3}\\
&= \lim_{x\to 0^+}\frac{1}{\rho(x)}\cdot\lim_{x\to 0^+}\frac{8\int_0^x(1+\sec t)\ln\sec t\,dt}{[g(x)]^3}\\
&= \frac{1}{3}.
\end{aligned}$$

(ii) When $x\to\frac{\pi}{2}^-$, x would be viewed as any number greater than $\frac{49\pi}{100}$, then $\ln\sec t>1$ for $\frac{\pi}{2}>t>\frac{49\pi}{100}$. For $\frac{\pi}{2}>t>\frac{49\pi}{100}$,

$$\begin{aligned}
&\int_0^x(1+\sec t)\ln\sec t\,dt\\
=\;&\int_{\cos^{-1}\frac{1}{e}}^x(1+\sec t)\ln\sec t\,dt+\int_0^{\cos^{-1}\frac{1}{e}}(1+\sec t)\ln\sec t\,dt\\
>\;&\int_{\cos^{-1}\frac{1}{e}}^x(1+\sec t)dt\\
=\;&x+\ln(\sec x+\tan x)-\cos^{-1}\frac{1}{e}+\ln(\sec(\cos^{-1}\frac{1}{e})+\tan\cos^{-1}(\frac{1}{e}))\\
\to\;&\infty\quad\text{as}\quad x\to\frac{\pi}{2}^-.
\end{aligned}$$

Using $\lim_{x\to\frac{\pi}{2}^-}\frac{\ln\sec x}{g(x)}=\lim_{x\to\frac{\pi}{2}^-}\frac{\tan x}{1+\sec x}=1$, we get

$$\begin{aligned}\lim_{x\to\frac{\pi}{2}^-}\sigma(x) &= \lim_{x\to\frac{\pi}{2}^-}\frac{\int_0^x(1+\sec t)\ln\sec t\,dt}{g(x)\ln\sec x}\\ &= \lim_{x\to\frac{\pi}{2}^-}\frac{(1+\sec x)\ln\sec x}{g(x)\tan x+(1+\sec x)\ln\sec x}\\ &= \lim_{x\to\frac{\pi}{2}^-}\frac{1}{\frac{g(x)}{\ln\sec x}\frac{\tan x}{1+\sec x}+1}\\ &= \frac{1}{2}.\end{aligned}$$

(iii) A. We need to show that the derivative is non-negative:

$$\begin{aligned}&(1-\csc x\cot x\ln\sec x)'\\ =\ & \csc x\cot^2 x\ln\sec x+\csc^3 x\ln\sec x-\csc x\cot x\tan x\\ =\ & \csc x(\cot^2 x\ln\sec x+\csc^2 x\ln\sec x-1).\end{aligned}$$

To achieve that, we observe that for $x>0$,

$$\cot^2 x\ln\sec x+\csc^2 x\ln\sec x-1\quad\text{is increasing,}$$

as, by (a),

$$\begin{aligned}&(\cot^2 x\ln\sec x+\csc^2 x\ln\sec x-1)'\\ =\ & ((\csc^2-1)\ln\sec x+\csc^2 x\ln\sec x-1)'\\ =\ & (2\csc^2 x\ln\sec x-\ln\sec x-1)'\\ =\ & -4\csc^2 x\cot x\ln\sec x+2\csc^2 x\cot x-\tan x\\ \ge\ & -4\csc^2 x\cot x(\frac{1}{2}\sin x\tan x)+2\csc^2 x\cot x-\tan x\\ =\ & \frac{-2\cos x+2-\sin^2 x}{\sin x\cos x}\\ =\ & \frac{(\cos x-1)^2}{\sin x\cos x}\\ \ge\ & 0,\end{aligned}$$

then, $\lim_{x\to 0^+}\frac{\ln\sec x}{\sin^2 x}=\lim_{x\to 0^+}\frac{\tan x}{2\sin x\cos x}=\frac{1}{2}$ gives us that

$$\begin{aligned}&\cot^2 x\ln\sec x+\csc^2 x\ln\sec x-1\\ \ge\ & \lim_{x\to 0^+}(\cot^2 x\ln\sec x+\csc^2 x\ln\sec x-1)\\ =\ & 0.\end{aligned}$$

Hence

$$1-\csc x\cot x\ln\sec x\quad\text{is increasing}$$

because $(1-\csc x\cot x\ln\sec x)'\ge 0$.

B. Note that

$$\frac{(1+\sec x)\ln\sec x}{\tan x\int_0^x 1+\sec t dt}=\frac{(\cot x+\csc x)\ln\sec x}{\int_0^x 1+\sec t dt},$$

$$\begin{aligned}&\frac{((\cot x+\csc x)\ln\sec x)'}{(\int_0^x 1+\sec t dt)'}\\ =&\ \frac{(\cot x+\csc x)\tan x-\csc x(\cot x+\csc x)\ln\sec x}{1+\sec x}\\ =&\ 1-\csc x\cot x\ln\sec x\\ \geq&\ \lim_{x\to 0^+}(1-\csc x\cot x\ln\sec x)\\ =&\ \frac{1}{2},\end{aligned}$$

and $\lim_{x\to 0^+}\frac{\ln\sec x}{\sin x}=\lim_{x\to 0^+}\frac{\tan x}{\cos x}=0$. Since

$$\lim_{x\to 0^+}((\cot x+\csc x)\ln\sec x)=0\quad\text{and}\quad\lim_{x\to 0^+}\int_0^x 1+\sec t dt=0,$$

and by the previous result, $1-\csc x\cot x\ln\sec x$ is positive and increasing, by the given fact we get $\frac{(1+\sec x)\ln\sec x}{\tan x\int_0^x 1+\sec t dt}$ is increasing.

C. $\lim_{x\to 0^+}\frac{\int_0^x 1+\sec t dt}{\sin x}=\lim_{x\to 0^+}\frac{1+\sec x}{\cos x}=2$ gives, by (b)(iii),

$$\begin{aligned}&\lim_{x\to 0^+}\frac{(1+\sec x)\ln\sec x}{\tan x\int_0^x 1+\sec t dt}\\ =&\ \lim_{x\to 0^+}\left((1+\cos x)\cdot\frac{\int_0^x 1+\sec t dt}{\sin x}\cdot\frac{\rho(x)}{8}\right)\\ =&\ \frac{1}{2}.\end{aligned}$$

D. Note that

$$\lim_{x\to 0^+}\int_0^x(1+\sec t)\ln\sec t dt=0\quad\text{and}\quad\lim_{x\to 0^+}(\ln\sec x\int_0^x 1+\sec t dt)=0,$$

and

$$\begin{aligned}&\frac{(\int_0^x(1+\sec t)\ln\sec t dt)'}{(\ln\sec x\int_0^x 1+\sec t dt)'}\\ =&\ \frac{(1+\sec x)\ln\sec x}{(1+\sec x)\ln\sec x+\tan x\int_0^x 1+\sec t dt}\\ =&\ \frac{1}{1+\frac{\tan x\int_0^x 1+\sec t dt}{(1+\sec x)\ln\sec x}}\end{aligned}$$

which implies that it is increasing by the previous result. And

$$\begin{aligned}\frac{(\int_0^x (1+\sec t)\ln\sec t dt)'}{(\ln\sec x\int_0^x 1+\sec t dt)'} &= \frac{1}{1+\frac{\tan x\int_0^x 1+\sec t dt}{(1+\sec x)\ln\sec x}} \\ &\geq \lim_{x\to 0^+}\frac{1}{1+\frac{\tan x\int_0^x 1+\sec t dt}{(1+\sec x)\ln\sec x}} \\ &= \frac{1}{3}\end{aligned}$$

which says that it is positive. Using the given fact, one obtains that $\sigma(x)$ is increasing.

□

8.3 Reduction formulae

Exercise 11. (Level 4)

For any non-negative integer n, let

$$I_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x \sin x \cos^n x dx.$$

(a) Evaluate I_0 and I_1.

(b) Show that for $n \geq 2$

$$\left[(n+1)^2+1\right] I_n = n(n-1)I_{n-2}.$$

(c) Show that when n is odd

$$I_n = \frac{n!}{[(n+1)^2+1]\cdots[2^2+1]}(e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}})$$

and when n is even

$$I_n = \frac{n!}{[(n+1)^2+1]\cdots[1^2+1]}(e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}}).$$

Solution. (a) We first note that

$$\begin{aligned}
I_0 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x \sin x dx \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x de^x \\
&= [e^x \sin x]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x d \sin x \\
&= e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x \cos x dx \\
&= e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x de^x \\
&= e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}} - [e^x \cos x]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x \sin x dx \\
&= e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}} - I_0.
\end{aligned}$$

It gives

$$I_0 = \frac{e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}}}{1^2 + 1}.$$

By several times of integration by parts, we get

$$\begin{aligned}
I_1 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x \sin x \cos x dx \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \cos x de^x \\
&= [e^x \sin x \cos x]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x d(\sin x \cos x) \\
&= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x (\cos^2 x - \sin^2 x) dx \\
&= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos^2 x - \sin^2 x) de^x \\
&= -\left[e^x (\cos^2 x - \sin^2 x)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x d(\cos^2 x - \sin^2 x) \\
&= e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}} - 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x \sin x \cos x dx \\
&= e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}} - 4I_1.
\end{aligned}$$

Hence, one obtains that

$$I_0 = \frac{e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}}{2^2 + 1}.$$

(Remark: The calculation of I_1 will be much simpler, if double angle formulae are used.)

(b) Integration by parts, we get

$$\begin{aligned}
I_n &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x \sin x \cos^n x dx \\
&= -\frac{1}{n+1} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x \sin x d \cos^{n+1} x \\
&= \frac{1}{n+1} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x \sin x \cos^{n+1} x dx \\
&= \frac{I_{n+1}}{n+1}.
\end{aligned}$$

On the other hand, again, integration by parts tells us that

$$\begin{aligned}
I_n &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x \sin x \cos^n x dx \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \cos^n x de^x \\
&= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x d(\sin x \cos^n x) \\
&= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x (\cos^{n+1} x - n \sin^2 x \cos^{n-1} x) dx \\
&= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x ((n+1) \cos^{n+1} x - n \cos^{n-1} x) dx \\
&= -(n+1) I_{n+1} + n I_{n-1}.
\end{aligned}$$

Using the preceeding two equations, we have, for $n \geq 2$,

$$I_n = -(n+1)I_{n+1} + nI_{n-1} = -(n+1)^2 I_n + n(n-1)I_{n-2},$$

that is,

$$\left[(n+1)^2 + 1\right] I_n = n(n-1)I_{n-2}.$$

(c) By (b), we have, for $n \geq 2$,

$$\begin{aligned}
I_n &= \frac{n(n-1)}{(n+1)^2+1} I_{n-2} \\
&= \frac{n(n-1)}{(n+1)^2+1} \cdot \frac{(n-2)(n-3)}{(n-1)^2+1} I_{n-4} \\
&\vdots \\
&= \begin{cases} \frac{n!}{[(n+1)^2+1]\cdots[2^2+1]} (e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}) & \text{when } n \text{ is odd} \\ \frac{n!}{[(n+1)^2+1]\cdots[1^2+1]} (e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}}) & \text{when } n \text{ is even.} \end{cases}
\end{aligned}$$

□

8.4 Improper integrals

Exercise 12. (Level 2/Level 3)
Discuss the convergence of the improper integrals below.

(a) $\int_0^{\infty} \frac{dx}{(1+x)(2+3x)}$

(b) $\int_2^\infty \frac{dx}{x(\ln x)^k}$

(c) $\int_2^\infty \frac{1-\ln x}{x^2}dx$

(d) $\int_0^1 3x^2 \ln x dx$

Solution. (a) Note that $\frac{1}{(1+x)(2+3x)} = \frac{-1}{1+x} + \frac{3}{2+3x}$. Since

$$\begin{aligned} \int_0^\infty \frac{dx}{(1+x)(2+3x)} &= \lim_{X\to\infty} \int_0^X \frac{dx}{(1+x)(2+3x)} \\ &= \lim_{X\to\infty} \int_0^X \frac{-1}{1+x} + \frac{3}{2+3x}dx \\ &= \lim_{X\to\infty} \left[\ln \frac{2+3x}{1+x}\right]_0^X \\ &= \lim_{X\to\infty} \left[\ln \frac{2+3X}{1+X} - \ln 2\right] \\ &= \ln \frac{3}{2}, \end{aligned}$$

the integral is convergent.

(b) We examine the convergence for the following three cases:

$k = 1$: Since

$$\int_2^\infty \frac{dx}{x\ln x} = \lim_{X\to\infty} \int_2^X \frac{dx}{x\ln x} = \lim_{X\to\infty} [\ln\ln x]_2^X = \lim_{X\to\infty} [\ln\ln X - \ln\ln 2] = \infty,$$

the integral is divergent.

$k < 1$: Since

$$\int_2^\infty \frac{dx}{x(\ln x)^k} = \lim_{X\to\infty} \int_2^X \frac{d\ln x}{(\ln x)^k} = \frac{1}{1-k} \lim_{X\to\infty} \left[(\ln x)^{1-k}\right]_2^X = \infty,$$

the integral is divergent.

$k > 1$: Since

$$\int_2^\infty \frac{dx}{x(\ln x)^k} = \lim_{X\to\infty} \int_2^X \frac{d\ln x}{(\ln x)^k} = \frac{1}{1-k} \lim_{X\to\infty} \left[(\ln x)^{1-k}\right]_2^X = \frac{(\ln 2)^{1-k}}{k-1},$$

the integral is convergent.

(c) Because

$$\begin{aligned} \int_2^X \frac{1-\ln x}{x^2}dx &= \int_2^X \frac{1}{x^2}dx + \int_2^X \ln x d\frac{1}{x} \\ &= \int_2^X \frac{1}{x^2}dx + \left[\frac{\ln x}{x}\right]_2^X - \int_2^X \frac{1}{x^2}dx \\ &= \frac{\ln X}{X} - \frac{\ln 2}{2} \end{aligned}$$

and by l'Hopital's rule, $\lim\limits_{X\to\infty}\frac{\ln X}{X}=\lim\limits_{X\to\infty}\frac{1}{X}=0$, therefore,

$$\int_2^{\infty}\frac{1-\ln x}{x^2}dx=-\frac{\ln 2}{2}.$$

The integral is convergent.

(d) Apply integration by parts on $\int 3x^2\ln x dx$:

$$\int 3x^2\ln x dx=\int \ln x dx^3=x^3\ln x-\int x^2 dx=x^3\ln x-\frac{x^3}{3}.$$

Consequently,

$$\begin{aligned}\int_0^1 3x^2\ln x dx &= \lim_{\varepsilon\to 0^+}\int_{\varepsilon}^1 3x^2\ln x dx\\ &= \lim_{\varepsilon\to 0^+}\left[x^3\ln x-\frac{x^3}{3}\right]_{\varepsilon}^1\\ &= \lim_{\varepsilon\to 0^+}\left[-\frac{1}{3}+\varepsilon^3\ln\varepsilon-\frac{\varepsilon^3}{3}\right]\\ &= -\frac{1}{3}.\end{aligned}$$

The last equality is justified by the limit (obtained by L'hopital's rule),

$$\lim_{\varepsilon\to 0^+}\varepsilon^3\ln\varepsilon=\lim_{\varepsilon\to 0^+}\frac{\ln\varepsilon}{\varepsilon^{-3}}=\lim_{\varepsilon\to 0^+}\frac{1}{-3\varepsilon^{-3}}=0.$$

□

Exercise 13. (Level 3/level 4)

Let $f(x)=\frac{(x+1)^2(x-1)}{x^3(x-2)}$. Find $\int_{-1}^3\frac{f'(x)}{1+f^2(x)}dx$.

Solution. The integrand $f(x)=\frac{(x+1)^2(x-1)}{x^3(x-2)}$ is unbounded at $x=0$ and $x=2$. We first treat the following four improper integrals:

$$\int_{-1}^0\frac{f'(x)}{1+f^2(x)}dx,\quad \int_0^1\frac{f'(x)}{1+f^2(x)}dx,\quad \int_1^2\frac{f'(x)}{1+f^2(x)}dx\quad\text{and}\quad \int_2^3\frac{f'(x)}{1+f^2(x)}dx.$$

Using $\int \frac{f'(x)}{1+f^2(x)}dx = \tan^{-1} f(x) + C$, we get

$$\begin{aligned}
\int_{-1}^{0} \frac{f'(x)}{1+f^2(x)}dx &= \lim_{a\to 0^-} \int_{-1}^{a} \frac{f'(x)}{1+f^2(x)}dx \\
&= \lim_{a\to 0^-} \left[\tan^{-1} f(x)\right]_{-1}^{a} \\
&= \lim_{a\to 0^-} \tan^{-1} f(a) \\
&= -\frac{\pi}{2}, \\
\int_{0}^{1} \frac{f'(x)}{1+f^2(x)}dx &= \lim_{b\to 0^+} \int_{b}^{1} \frac{f'(x)}{1+f^2(x)}dx \\
&= \lim_{b\to 0^+} \left[\tan^{-1} f(x)\right]_{b}^{1} \\
&= \lim_{b\to 0^+} -\tan^{-1} f(b) \\
&= -\frac{\pi}{2}, \\
\int_{1}^{2} \frac{f'(x)}{1+f^2(x)}dx &= \lim_{c\to 2^-} \int_{1}^{c} \frac{f'(x)}{1+f^2(x)}dx \\
&= \lim_{c\to 2^-} \left[\tan^{-1} f(x)\right]_{1}^{c} \\
&= \lim_{c\to 2^-} \tan^{-1} f(c) \\
&= -\frac{\pi}{2}, \quad \text{and} \\
\int_{2}^{3} \frac{f'(x)}{1+f^2(x)}dx &= \lim_{d\to 2^+} \int_{d}^{3} \frac{f'(x)}{1+f^2(x)}dx \\
&= \lim_{d\to 2^+} \left[\tan^{-1} f(x)\right]_{d}^{3} \\
&= \lim_{d\to 2^+} \left(\tan^{-1} \frac{32}{27} - \tan^{-1} f(d)\right) \\
&= \tan^{-1} \frac{32}{27} - \frac{\pi}{2}.
\end{aligned}$$

Therefore, we arrive at

$$\begin{aligned}
&\int_{-1}^{3} \frac{f'(x)}{1+f^2(x)}dx \\
= \ &\int_{-1}^{0} \frac{f'(x)}{1+f^2(x)}dx + \int_{0}^{1} \frac{f'(x)}{1+f^2(x)}dx + \int_{1}^{2} \frac{f'(x)}{1+f^2(x)}dx + \int_{2}^{3} \frac{f'(x)}{1+f^2(x)}dx \\
= \ &-\frac{\pi}{2} - \frac{\pi}{2} - \frac{\pi}{2} + \tan^{-1} \frac{32}{27} - \frac{\pi}{2} \\
= \ &\tan^{-1} \frac{32}{27} - 2\pi.
\end{aligned}$$

$\square$

8.5 Inequalities Involving Integrals

Exercise 14. (Level 2)
By comparing $1+2x$, $1+x$ and $1+\frac{x}{2}$, show that, for $x \geq 1$

$$\ln\sqrt{1+2x} \leq \ln(1+x) \leq \ln(1+\frac{x}{2})^2.$$

Solution. Observe that for $t \geq 0$,

$$\frac{1}{1+2t} \leq \frac{1}{1+t} \leq \frac{1}{1+\frac{t}{2}}.$$

Integrating from 0 to x, it gives

$$\int_0^x \frac{1}{1+2t}dt \leq \int_0^x \frac{1}{1+t}dt \leq \int_0^x \frac{1}{1+\frac{t}{2}}dt$$

which says that

$$\ln\sqrt{1+2x} \leq \ln(1+x) \leq \ln(1+\frac{x}{2})^2.$$

□

Exercise 15. (Level 3)
Suppose that h is positive and continuous on $[0,\infty)$ and that $h(x) \geq H(x)$, for $x > 0$, where

$$H(x) = 1 + \int_0^x h(t)dt.$$

Prove that, for any $x > 0$,

$$h(x) \geq e^x.$$

Solution. Since h is positive, H is positive. By the fundamental theorem of calculus, $H'(t) = h(t) \geq H(t)$, we then integrate from 0 to x,

$$\int_0^x \frac{H'(t)}{H(t)}dt \geq \int_0^x dt.$$

With $H(0) = 1$, we have $\ln H(x) \geq x$, i.e.

$$h(x) \geq H(x) \geq e^x.$$

□

Exercise 16. (Level 3)
Let x be a real number. Express $\ln(1+x^2)$ in integral form and, considering this integral, show that, when $n > 0$,

$$\frac{nx^2}{1=x^2} < \ln(1+x^2)^n < nx^2.$$

Solution. Observe that $\ln(1+x^2) = \int_1^{1+x^2} \frac{1}{t} dt$ and for $1 \le t \le 1+x^2$,

$$\frac{1}{1+x^2} \le \frac{1}{t} \le 1.$$

Therefore,

$$\ln(1+x^2)^n = n \int_1^{1+x^2} \frac{1}{t} dt \ge n \int_1^{1+x^2} \frac{1}{1+x^2} dt = \frac{nx^2}{1+x^2}$$

and

$$\ln(1+x^2)^n = n \int_1^{1+x^2} \frac{1}{t} dt \le n \int_1^{1+x^2} 1 dt = nx^2.$$

Combining, we get

$$\frac{nx^2}{1=x^2} < \ln(1+x^2)^n < nx^2,$$

as desired.

□

Exercise 17. (Level 5)

(a) Show that, for any real numbers a and x,

$$(1-|a|)^2 \le 1 - 2a\cos x + a^2 \le (1+|a|)^2.$$

(b) For any real number a such that $|a| \ne 1$, let

$$I(a) = \int_0^{\pi} \ln(1 - 2a\cos x + a^2) dx.$$

(i) By using (a), show that

$$\pi(1-|a|)^2 \le I(a) \le \pi(1+|a|)^2,$$

and deduce that $\lim_{a\to 0} I(a) = 0$.

(ii) Using $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$, show that

$$I(a) + I(-a) = I(a^2) \quad \text{and} \quad I(a) = I(-a).$$

Hence show that

$$I(a) = \frac{1}{2^n} I(a^{2^n})$$

for all positive integer n.

(iii) Show that

$$I(\frac{1}{a}) = I(a) - 2\pi \ln |a|.$$

(iv) Deduce from above results that

$$I(a) = \begin{cases} 0 & \text{if } |a| < 1 \\ 2\pi \ln |a| & \text{if } |a| > 1 \end{cases}$$

(c) Show that $I(1) = I(-1) = 0$.

Solution. (a) We begin with

$$-2|a| \leq -2a\cos x \leq 2|a|,$$

then we get

$$(1-|a|)^2 \leq 1 - 2a\cos x + a^2 \leq (1+|a|)^2.$$

(b) (i) Using the inequalities derived in (a),

$$\int_0^\pi (1-|a|)^2 dx \leq \int_0^\pi \ln(1-2a\cos x + a^2)dx \leq \int_0^\pi (1+|a|)^2 dx,$$

i.e.

$$\pi(1-|a|)^2 \leq I(a) \leq \pi(1+|a|)^2.$$

As $\lim_{a\to 0}(1 \pm |a|)^2 = 0$, by the sandwich theorem, we get $\lim_{a\to 0} I(a) = 0$.

(ii) We first have that

$$\begin{aligned}
& I(a) + I(-a) \\
= \;& \int_0^\pi \ln(1-2a\cos x + a^2)dx + \int_0^\pi \ln(1+2a\cos x + a^2)dx \\
= \;& \int_0^\pi \ln(1-2a^2\cos(2x) + a^4)dx \\
= \;& \frac{1}{2}\int_0^{2\pi} \ln(1-2a^2\cos t + a^4)dt \\
= \;& \frac{1}{2}\int_0^{\pi} \ln(1-2a^2\cos t + a^4)dt + \frac{1}{2}\int_\pi^{2\pi} \ln(1-2a^2\cos t + a^4)dt \\
= \;& \frac{1}{2}I(a^2) - \frac{1}{2}\int_\pi^{2\pi} \ln(1-2a^2\cos(2\pi - t) + a^4)d(2\pi - t) \\
= \;& \frac{1}{2}I(a^2) + \frac{1}{2}\int_0^{\pi} \ln(1-2a^2\cos s + a^4)ds \\
= \;& \frac{1}{2}I(a^2) + \frac{1}{2}I(a^2) \\
= \;& I(a^2)
\end{aligned}$$

and

$$\begin{aligned}
I(-a) &= \int_0^\pi \ln(1-2a\cos x + a^2)dx \\
&= -\int_0^\pi \ln(1+2a\cos(\pi - x) + a^2)d(\pi - x) \\
&= \int_0^\pi \ln(1+2a\cos t + a^2)dt \\
&= I(a).
\end{aligned}$$

Hence, inductively, we get

$$I(a) = \frac{1}{2} I(a^2) = \cdots = \frac{1}{2^n} I(a^{2^n}).$$

(iii)

$$\begin{aligned} I(\frac{1}{a}) &= \int_0^\pi \ln(1 - \frac{2}{a}\cos x + \frac{1}{a^2})dx \\ &= \int_0^\pi \ln(1 - 2a\cos x + a^2)dx - \int_0^\pi \ln a^2 dx \\ &= I(a) - 2\pi \ln |a|. \end{aligned}$$

(iv) When $|a| < 1$, $\lim_{n\to\infty} a^{2^n} = 0$, by (ii) and (i)

$$I(a) = \lim_{n\to\infty} \frac{1}{2^n} I(a^{2^n}) = 0.$$

When $|a| > 1$, we use the result in (iii), we then have

$$I(a) = I(\frac{1}{a}) + 2\pi \ln |a| = 2\pi \ln |a|.$$

The last equability follows from $I(\frac{1}{a}) = 0$ as $\frac{1}{|a|} < 1$.

(c) The derivations of $I(a) + I(-a) = I(a^2)$ and $I(a) = I(-a)$ haven't used the assumption $|a| \neq 1$, the results remain true when $a = \pm 1$. Hence solving $I(1) + I(-1) = I(1)$ and $I(1) = I(-1)$, we have $I(1) = I(-1) = 0$, as desired.

□

Exercise 18. (Level 5)

(a) Suppose $f(x), g(x)$ are continuously differentiable functions such that $f'(x) \geq 0$ for $a \leq x \leq b$.

(i) Let $w(x) = \int_a^x g(t)dt$. Show that

$$\int_a^b f(x)g(x)dx = f(b)w(b) - \int_a^b f'(x)w(x)dx.$$

(ii) Using the mean value theorem for integral, show that

$$\int_a^b f(x)g(x)dx = f(b)\int_c^b g(x)dx + f(a)\int_a^c g(x)dx$$

for $c \in [a, b]$.

(b) Let $F(x)$ be a function with a continuous second derivative such that $F''(x) \geq 0$ and $F'(x) \geq m > 0$ for $a \leq x \leq b$. Using (a) with $f(x) = -\frac{1}{F'(x)}$ and $g(x) = -F'(x)\cos F(x)$. Show that

$$\left| \int_a^b \cos F(x) dx \right| \leq \frac{4}{m}.$$

(Remark: This lemma plays an important role in the theory of exponential sums.)

(c) (i) Show that

$$\int_0^1 \cos x^{n+1} dx \leq \int_0^1 \cos x^n dx.$$

Hence show that $\lim_{n\to\infty} \int_0^1 \cos x^n dx$ exists.

(ii) Using (b), or otherwise, show that $\lim_{n\to\infty} \int_0^{2\pi} \cos x^n dx$ exists.

Solution. (a) (i) Note that $w(a) = 0$, we get

$$\int_a^b f(x)g(x)dx = \int_a^b f(x)dw(x) = f(b)w(b) - \int_a^b f'(x)w(x)dx.$$

(ii) By the mean value theorem for integral, there is a $c \in [a, b]$ such that

$$\int_a^b f'(x)w(x)dx = w(c)\int_a^b f'(x)dx = f(b)w(c) - f(a)w(c).$$

Plugging in the equation of (i), one obtains

$$\begin{aligned}
\int_a^b f(x)g(x)dx &= f(b)w(b) - \int_a^b f'(x)w(x)dx \\
&= f(b)w(b) - w(c)\int_a^b f'(x)dx \\
&= f(b)w(b) - f(b)w(c) + f(a)w(c) \\
&= f(b)\int_a^b g(x)dx - f(b)\int_a^c g(x)dx + f(a)\int_a^c g(x)dx \\
&= f(b)\int_c^b g(x)dx + f(a)\int_a^c g(x)dx.
\end{aligned}$$

(b) Note that

$$|f(a)| = |\frac{1}{F'(a)}| \leq \frac{1}{m},$$

$$|f(b)| \leq \frac{1}{m},$$

$$|\int_c^b g(x)dx| = |\int_c^b F'(x)\cos F(x)dx| = |\int_c^b \cos F(x)dF(x)| = |\sin F(b) - \sin F(c)| \leq 2,$$

and

$$|\int_a^c g(x)dx| \leq 2.$$

$F''(x) \geq 0$ ensures that $f'(x) = \frac{F''(x)}{(F'(x))^2} \geq 0$. We are ready to use (a) and the estimations above. Then

$$\begin{aligned}\left|\int_a^b \cos F(x)dx\right| &= |f(b)\int_c^b g(x)dx + f(a)\int_a^c g(x)dx| \\ &\leq |f(b)\int_c^b g(x)dx| + |f(a)\int_a^c g(x)dx| \\ &\leq \frac{2}{m} + \frac{2}{m} \\ &\leq \frac{4}{m}.\end{aligned}$$

(c) (i) For $x \in [0,1]$, $x^n \geq x^{n+1}$, then $\cos x^n \leq \cos x^{n+1} \leq 1$ so, by integrating from 0 to 1,

$$\int_0^1 \cos x^n dx \leq \int_0^1 \cos x^{n+1} dx \leq 1.$$

Hence the sequence $\{\int_0^1 \cos x^n dx\}$ is increasing and bounded by 1, by monotone convergence theorem, $\lim_{n\to\infty} \int_0^1 \cos x^n dx$ exists.

(ii) For $1 \leq x \leq 2\pi$, let $F(x) = x^n$. Then $F'(x) = nx^{n-1} \geq n$ and $F''(x) = n(n-1)x^{n-2} \geq 0$ for $1 \leq x \leq 2\pi$. We are in position to use the result of (b), we get

$$|\int_1^{2\pi} \cos x^n dx| \leq \frac{4}{n}.$$

By the sandwich theorem tells us that

$$\lim_{n\to\infty} \int_1^{2\pi} \cos x^n dx = 0.$$

Hence, by (c)(i), $\lim_{n\to\infty} \int_0^{2\pi} \cos x^n dx = \lim_{n\to\infty} \int_0^1 \cos x^n dx + \lim_{n\to\infty} \int_1^{2\pi} \cos x^n dx = \lim_{n\to\infty} \int_0^1 \cos x^n dx$ exists.

□