THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics (Spring 2018)
Tutorial 8
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Taylor's Theorem

Let f be a function that is $k+1$ times differentiable on (a,b). Let $x_0 \in (a,b)$.
Let $x \in (a,b)$. There exists ξ between x and x_0 such that

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + ... + \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(k+1)}(\xi)}{(k+1)!}(x-x_0)^{k+1}$$

Examples of Taylor's Polynomial

The followings are Taylor Series of some common function centered $x=0$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + ... = \sum_{n=0}^{\infty} x^n \qquad \text{for } x \in (-1,1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} ... = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{for } x \in (-\infty,\infty)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n!} \qquad \text{for } x \in (-\infty,\infty)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \text{for } x \in (-\infty,\infty)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \qquad \text{for } x \in (-1,1]$$

Exercise 1 (Continuation of Ex4 from last tutorial):

(a) Write down the Taylor polynomial $P_3(x)$, where $f(x) = \ln(1-x)$ centered at 0.

(b) Hence, approximate $\ln 0.99$.

(c) Show that the error of the approximation in (b) is less than 10^{-7}.

Exercise 2:

Find the third Taylor polynomial of

(a) $f(x) = e^x \sin x, \quad x_0 = 0$ (b) $f(x) = \ln(1+\sin x), \quad x_0 = 0$

Exercise 3:

Let

$$f(x) = x^2 \cos x$$

(a) Find the Taylor series of f centered at 0.

(b) Find $f^{(99)}(0)$ and $f^{(100)}(0)$.

Exercise 4:

By Taylor's theorem or L'Hopital's rule , evaluate

(a) $\displaystyle\lim_{x\to 0} \frac{\ln(1+x^2)}{x \sin x}$ (b) $\displaystyle\lim_{x\to 0} \frac{1}{\ln(1+x)} - \frac{1}{x}$

Exercise 5:

Show that for all $x > 0$,

$$1 + \frac{x}{2} - \frac{x^2}{8} \leq \sqrt{1+x} \leq 1 + \frac{x}{2}$$

Solution

Exercise 1:

(a) Check the previous tutorial

(b) Check the previous tutorial

(c)

$$f^{(4)}(x) = -\frac{6}{(1-x)^4}$$

By Taylor's theorem, there exists $\xi \in (0, 0.01)$ such that

$$|f(0.01) - P_3(0.01)| = \frac{|f^{(4)}(\xi)|}{4!}|0.01|^4$$

$$|f^{(4)}(\xi)| = \left|\frac{6}{(1-\xi)^4}\right| = \frac{6}{|1-\xi|^4} \leq \frac{6}{0.99^4} \leq 6 \cdot 2^4$$

$$|f(0.01) - P_3(0.01)| \leq 6 \cdot 0.01^4 < 10 \cdot 10^{-8} = 10^{-7}$$

Exercise 2:

(a)

$$e^x \sin x = (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} ...)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) = x + x^2 + \frac{x^3}{3} + \cdots$$

The third order Taylor polynomial is

$$x + x^2 + \frac{x^3}{3}$$

(b)

$$\begin{aligned}
\ln(1 + \sin x) &= \sin x - \frac{(\sin x)^2}{2} + \frac{(\sin x)^3}{3} - \frac{(\sin x)^4}{4} + ... \\
&= (x - \frac{x^3}{3!} + ...) - \frac{(x - \frac{x^3}{3!} + ...)^2}{2} + \frac{(x - \frac{x^3}{3!} + ...)^3}{3} + ... \\
&= x - \frac{x^2}{2} + \frac{x^3}{6} + ...
\end{aligned}$$

The third order Taylor polynomial is

$$x - \frac{x^2}{2} + \frac{x^3}{6}$$

Exercise 3:

(a) The Taylor series of f centered at 0

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{2n!}$$

(b) The series of f centered at 0 is

$$f(x) = \sum_{n=0}^{\infty} f^{(m)}(0) \frac{x^m}{m!}$$

Now,

$$f(x) = \sum_{n=0}^{\infty} (-1)^n (2n+2)(2n+1) \frac{x^{2n+2}}{(2n+2)!}$$

If $m = 2n+2$,

$$f^{(m)}(0) = (-1)^n(2n+2)(2n+1)$$

Set $m = 99$. There is no n that satisfies $m = 2n+2$. Hence, $f^{(99)}(0) = 0$.
Set $m = 100$. Then $n = 49$, and $f^{(100)}(0) = (-1)(100)(99) = -9900$.

Exercise 4:

(a) The 2-th Taylor polynomial of $\ln(1+x^2)$ centered at 0 is x^2.
The 2-nd Taylor polynomial of $x \sin x$ centered at 0 is x^2.

By Taylor's theorem, there exist constants C, D such that

$$\ln(1+x^2) = x^2 + Cx^3$$

$$x \sin x = x(x + Dx^2) = x^2 + Dx^3$$

$$\lim_{x \to 0} \frac{\ln(1+x^2)}{x \sin x} = \lim_{x \to 0} \frac{x^2 + Cx^3}{x^2 + Dx^3} = \lim_{x \to 0} \frac{1 + Cx}{1 + Dx} = 1$$

(b) The 2-nd Taylor polynomial of $\ln(1+x)$ centered at 0 is $x - \frac{x^2}{2}$.

By Taylor's theorem, there exist a constant C such that

$$\ln(1+x) = x - \frac{x^2}{2} + Cx^3$$

$$\lim_{x \to 0} \frac{1}{\ln(1+x)} - \frac{1}{x} = \lim_{x \to 0} \frac{1}{x - \frac{x^2}{2} + Cx^3} - \frac{1}{x} = \lim_{x \to 0} \frac{\frac{x^2}{2} - Cx^3}{x(x - \frac{x^2}{2} + Cx^3)} = \lim_{x \to 0} \frac{\frac{1}{2} - Cx}{1 - \frac{x}{2} + Cx^2} = \frac{1}{2}$$

Exercise 5:

$$f'(x) = \frac{1}{2(1+x)^{\frac{1}{2}}}$$

$$f''(x) = -\frac{1}{4(1+x)^{\frac{3}{2}}}$$

$$f'''(x) = \frac{3}{8(1+x)^{\frac{5}{2}}}$$

$$f'(0) = \frac{1}{2}, f''(0) = -\frac{1}{4}$$

The taylor polynomial of degree 1 of f centered at 0 is

$$P_1(x) = 1 + \frac{x}{2}$$

The taylor polynomial of degree 2 of f centered at 0 is

$$P_2(x) = 1 + \frac{x}{2} - \frac{x^2}{8}$$

By taylor's theorem, there exists $\xi_1, \xi_2 \in (0, x)$ such that

$$f(x) - p_1(x) = \frac{f''(\xi_1)x^2}{2} = -\frac{x^2}{8(1+\xi_1)^{\frac{3}{2}}} \leq 0$$

$$f(x) - p_2(x) = \frac{f'''(\xi_2)x^3}{3!} = \frac{3x^3}{8(6)(1+\xi_2)^{\frac{5}{2}}} \geq 0$$

Hence,

$$1 + \frac{x}{2} - \frac{x^2}{8} = p_2(x) \leq f(x) \leq p_1(x) = 1 + \frac{x}{2}$$