THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics (Spring 2018)
Tutorial 2
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1. Mathematical Induction

The First Principle of Mathematical Induction

Let $P_n = P(n)$ be a proposition, or a statement involving the natural number n.
Let $n_0 \in \mathbb{N}$. Given P_{n_0} is true. Suppose for $k \in \mathbb{N}$, $k \geq n_0$, if P_k is true, then P_{k+1} is true.
Then P_n is true for all $n \in \mathbb{N}, n \geq n_0$.

We may view $\{P_n\}$ as a sequence of propositions: $P_n = 1$ if P_n is true; otherwise $P_n = 0$.
From the above principle, one can provide a definition for the sequence $\{P_n\}$ as

$$P_{n_0} = 1, \quad P_{n+1} = P_n \quad \text{for} \quad n \geq n_0$$

This shows that $P_n = 1$ for all $n \in \mathbb{N}$, $n \geq n_0$.

The Second Principle of Mathematical Induction (Optional)

Let $P_n = P(n)$ be a proposition, or a statement involving the natural number n. Let $n_0 \in \mathbb{N}$.
Given P_{n_0}, P_{n_0+1} are true. Suppose for $k \in \mathbb{N}$, $k \geq n_0$, if P_k, P_{k+1} are true, then P_{k+2} is true.
Then P_n is true for all $n \in \mathbb{N}, n \geq n_0$.

Can you figure out a definition for the sequence $\{P_n\}$?

2. Function

In the last tutorial, you should know that a sequence is an example of functions.
This tutorial mainly focuses on the definitions of some real-valued functions.

Definition

(a) **Domain and image of a function**
Let $f : D \subset \mathbb{R} \to \mathbb{R}$ be a function. Define $f(D) = \{f(x) : x \in D\}$.
The set D is the domain of f, and the set $f(D)$ is the image of f.

(b) **Injective Function**
Let $f : D \to \mathbb{R}$ be a function. The function f is said to be injective (one-to-one) if for any $x_1, x_2 \in D$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

(c) **Surjective Function**
Let $f : D \to \mathbb{R}$ be a function. The function f is said to be surjective (onto) if for any $y \in \mathbb{R}$, there exists $x \in D$ such that $f(x) = y$.

(d) **Bijective Function**
A function is said to be bijective if it is both injective and surjective.

(e) **Even Function**
Let $f : D \to \mathbb{R}$ be a function.
The function f is said to be even if for any $x \in D$, $f(-x) = f(x)$.

(f) **Odd Function**
Let $f : D \to \mathbb{R}$ be a function.
The function f is said to be odd if for any $x \in D$, $f(-x) = -f(x)$.

Exercise 1 (For students who are not familiar with mathematical induction or trigonometry):

(1) Prove the following for all $n \in \mathbb{N}$ by induction.

(a) $1 + 3 + 6 + ... + \dfrac{n(n+1)}{2} = \dfrac{n(n+1)(n+2)}{3!}$

(b) $1^3 + 2^3 + 3^3 + ... + (2n)^3 = n^2(2n+1)^2$

(c) $\sin\theta + \sin 3\theta + ... + \sin(2n-1)\theta = \dfrac{\sin^2 n\theta}{\sin\theta}$, where θ is not a multiple of 2π

(2) Let $\{a_n\}$ be the sequence defined by

$$a_0 = \sqrt{2}, \qquad a_n = \sqrt{2 + a_{n-1}}$$

Show, by induction, that $a_n = 2\cos\dfrac{\pi}{2^{n+2}}$ for all non-negative integer n.

Exercise 2:

Let f be a function. Find the domain and image of the function f.

(a) $f(x) = \dfrac{1}{\sqrt{x^2 - 9}}$ (b) $f(x) = \dfrac{1}{\sin x + \cos x}$

Exercise 3:

Let f be a function. Determine whether the function f is even or odd.

(a) $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}; \quad f(x) = \dfrac{1}{x^2}$

(b) $f : \mathbb{R} \to \mathbb{R}; \quad f(x) = \dfrac{\exp(x) - \exp(-x)}{2}$

Exercise 4:

Let f be a function. Determine whether the function f is injective, surjective and bijective.

(a) $f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}; \quad f(x) = \dfrac{1}{x}$

(b) $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}; \quad f(x) = \dfrac{1}{x}$

(c) $f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}; \quad f(x) = \dfrac{1}{|x|}$

(d) $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}^+; \quad f(x) = \dfrac{1}{|x|}$

(e) $f : \mathbb{R} \setminus \{2\} \to \mathbb{R}; \quad f(x) = \dfrac{3x + 2}{x - 2}$

Exercise 5:

Let f be a function. Sketch the graph of the function f.

(a) $f(x) = |x - 2| + 1$

(b) $f(x) = ||x - 3| - 6|$

Solution

Exercise 1:

(1) (a) Please verify it yourself.

(b) Let $P(n)$ be the proposition that $1^3+2^3+3^3+...+(2n)^3 = n^2(2n+1)^2$.
Since $1^3+2^3 = 1^2(2+1)^2$, $P(1)$ is true.
Suppose $P(k)$ is true for some $k \in \mathbb{N}$. i.e. $1^3+2^3+3^3+...+(2k)^3 = k^2(2k+1)^2$. Then

$$\begin{aligned}
&1^3+2^3+3^3+...+\left(2k\right)^3+\left(2k+1\right)^3+\left(2(k+1)\right)^3\\
&= k^2\left(2k+1\right)^2+\left(2k+1\right)^3+\left(2(k+1)\right)^3\\
&= \left(k^2+2k+1\right)\left(2k+1\right)^2+\left(2(k+1)\right)^3\\
&= \left(k+1\right)^2\left(2k+1\right)^2+8\left(k+1\right)^3\\
&= \left(k+1\right)^2\left(4k^2+12k+9\right)\\
&= \left(k+1\right)^2\left(2(k+1)+1\right)^2
\end{aligned}$$

Hence, $P(k+1)$ is true.
By the first principle of mathematical induction, $P(n)$ is true for any $n \in \mathbb{N}$.

(c) Hint: Apply the following identities:
(i) $2\sin x \sin y = \cos(x-y) - \cos(x+y)$
(ii) $\cos 2x = 1 - 2\sin^2 x$

(2) Hint: $\cos 2x = 2\cos^2 x - 1$

Exercise 2:

(a) Let $D(f), R(f)$ be the domain and image of f respectively.
Note that $\sqrt{x^2-9} > 0 \iff x > 3$ or $x < -3$.
Hence $D(f) = \left\{x \in \mathbb{R} \mid \sqrt{x^2-9} > 0\right\} = (-\infty, -3) \cup (3, +\infty)$.
Also, $R(f) = (0, +\infty)$.

(b) Let $D(f), R(f)$ be the domain and image of f respectively.
One has
$$\sin x + \cos x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$$
Note that $\sin x + \cos x \neq 0 \iff \sin\left(x+\frac{\pi}{4}\right) \neq 0 \iff x + \frac{\pi}{4} \neq n\pi$, where $n \in \mathbb{Z}$.
Hence $D(f) = \left\{x \in \mathbb{R} \mid \sin x + \cos x \neq 0\right\} = \left\{x \in \mathbb{R} \mid x \neq \frac{4n-1}{4}\pi, n \in \mathbb{Z}\right\}$.
Also, $R(f) = \left(-\infty, -\frac{1}{\sqrt{2}}\right] \cup \left[\frac{1}{\sqrt{2}}, +\infty\right)$.

Exercise 3:

(a) Even. Please verify it yourself.

(b) $f(-x) = \dfrac{\exp(-x) - \exp(x)}{2} = -f(x)$. So f is odd.
Remark: $f(x) = \sinh(x)$

Exercise 4:

Note that
$\mathbb{R} \setminus \{0\}$ is the set of all real numbers except 0;
$\mathbb{R}^+$ is the set of all positive real numbers.

question	**injective**	**surjective**	**bijective**
4a	$\surd$	$\surd$	$\surd$
4b	$\surd$	$\times$	$\times$
4c	$\times$	$\times$	$\times$
4d	$\times$	$\surd$	$\times$

(a-c) Please verify it yourself.

(d) Take $x = 1, y = -1$. Note that $x \neq y$ and $f(x) = f(y) = 1$.
Hence f is not injective.

Let $y \in \mathbb{R}^+$. Take $x = \dfrac{1}{y}$. Note that $x \in \mathbb{R} \setminus \{0\}$ and $f(x) = y$.
Hence f is surjective.

Therefore, f is not bijective.

(e) Let $f(x) = f(y)$. Then $\dfrac{3x+2}{x-2} = \dfrac{3y+2}{y-2}$. One has

$$(3x+2)(y-2) = (3y+2)(x-2)$$

$$3xy + 2y - 6x - 4 = 3xy + 2x - 6y - 4$$

$$8x = 8y$$

$$x = y$$

Hence f is injective.

Let $y = 3$. There is no such x that satisfies $f(x) = y$. For otherwise, contradiction arises:

$$3 = \frac{3x+2}{x-2}$$

$$3x - 6 = 3x + 2$$

$$0 = 8$$

Hence f is not surjective.

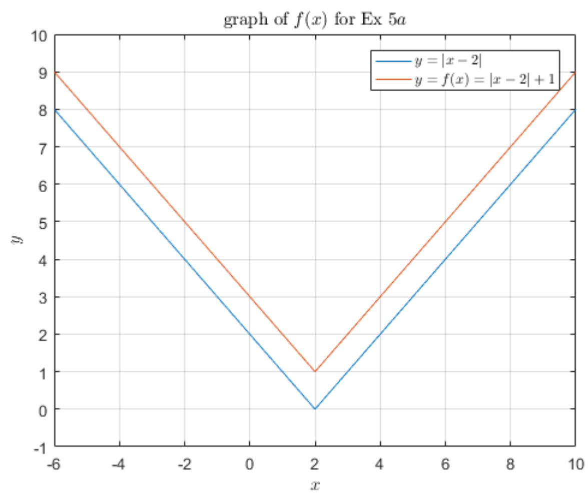
Therefore, f is not bijective.

Exercise 5:

(a) Let $g(x) = |x - 2|, \quad h(x) = 1$. We have $f(x) = g(x) + h(x)$.

$$g(x) = \begin{cases} x - 2 & x \geq 2 \\ 2 - x & x > 2 \end{cases}$$

$$f(x) = \begin{cases} x - 1 & x \geq 2 \\ 3 - x & x > 2 \end{cases}$$



(b) Let $g(x) = |x - 3|, \quad h(x) = -6$. We have $f(x) = |g(x) + h(x)|$.

$$g(x) = \begin{cases} x - 3 & x \geq 3 \\ 3 - x & x < 3 \end{cases}$$

$$g(x) + h(x) = \begin{cases} x - 9 & x \geq 3 \\ -3 - x & x < 3 \end{cases}$$

$$f(x) = \begin{cases} x - 9 & x \geq 9 \\ 9 - x & 3 \leq x < 9 \\ 3 + x & -3 \leq x < 3 \\ -3 - x & x < -3 \end{cases}$$