MATH 1010DE Week 1

Sequences

1.1 Sequences and Limits

A **sequence** is an ordered list of numbers:

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

Common notations:

$$\{a_n\},\ \{a_n\}_{n\in\mathbb{N}},\ \{a_n\}_{n=1}^{\infty}$$

Example 1.1.

- $$a_n = \sqrt{n}\,, \quad n \in \mathbb{N}$$
  $$\{a_n\}_{n\in\mathbb{N}} = \{1, \sqrt{2}, \sqrt{3}, \ldots\}.$$

- $$b_n = (-1)^{n+1}\frac{1}{n}\,, \quad n \in \mathbb{N}$$
  $$\{b_n\} = \left\{1, -\frac{1}{2}\,, \frac{1}{3}\,, -\frac{1}{4}\,, \ldots\right\}.$$

- **Fibonacci Sequence**

$$a_1 = 1, a_2 = 1$$

$$a_n = a_{n-2} + a_{n-1} \textit{ for } n \geq 3.$$

$$\{a_n\} = \{1, 1, 2, 3, 5, 8, 13, \ldots\}$$

  In this case we say that the sequence $\{a_n\}$ is defined **recursively** .

Sometimes, the terms a_n of a sequence approach a single value L as n tends to infinity.

Definition 1.2. *We say that the* **limit** *of a sequence $\{a_n\}$ is equal to L if for all real numbers $\varepsilon > 0$ the exists a number $N > 0$ such that $|a_n - L| < \varepsilon$ for all $n > N$.*

If such a number L exists, we say that: $\{a_n\}$ **converges** to L, and write:

$$\lim_{n\to\infty} a_n = L.$$

If no such L exists, we say that $\{a_n\}$ **diverges** .

If the values of a_n increase (resp. decrease) without bound, we say that $\{a_n\}$ diverges to ∞ (resp. $-\infty$), and write:

$$\lim_{n\to\infty} a_n = \infty \quad \text{(resp. } -\infty).$$

Exercise 1.3. *1.* **WeBWorK**

2. **WeBWorK**

3. **WeBWorK**

4. **WeBWorK**

1.1.1 Useful Properties

- **Constant sequence**

  If $a_n = c$ for all n, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c = c$.

- **Sum/Difference rule**

  If both $\{a_n\}$ and $\{b_n\}$ converge, then:

$$\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n.$$

- **Product Rule**

  If both $\{a_n\}$ and $\{b_n\}$ converge, then:

$$\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right) \cdot \left(\lim_{n\to\infty} b_n\right).$$

- **Quotient Rule**

  If both $\{a_n\}$ and $\{b_n\}$ converge, and $\lim_{n\to\infty} b_n \neq 0$, then:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}.$$

- $$\lim_{n\to\infty} \frac{1}{n} = 0.$$

- In general, if $\lim_{n\to\infty} a_n = +\infty$ or $\lim_{n\to\infty} a_n = -\infty$, we have:

$$\lim_{n\to\infty} \frac{1}{a_n} = 0.$$

1.1.2 Examples

- $\lim\limits_{n\to\infty} \dfrac{3n^2-2n+7}{2n^2+3}$

$$\begin{aligned} &= \lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n^2}} \cdot \frac{3n^2-2n+7}{2n^2+3} \\ &= \lim_{n\to\infty} \frac{3-\frac{2}{n}+\frac{7}{n^2}}{2+\frac{3}{n^2}} \\ &= \frac{3}{2}. \end{aligned}$$

- $\lim\limits_{n\to\infty} \dfrac{-3n^2}{\sqrt[3]{27n^6-5n+1}}$

$$\begin{aligned} &= \lim_{n\to\infty} \frac{-3n^2}{n^2\sqrt[3]{27-\frac{5}{n^5}+\frac{1}{n^6}}} \\ &= \lim_{n\to\infty} \frac{-3}{\sqrt[3]{27-\frac{5}{n^5}+\frac{1}{n^6}}} \\ &= \frac{-3}{\sqrt[3]{27}} = -1. \end{aligned}$$

- $\lim\limits_{n\to\infty} \sqrt{4n^2+n} - \sqrt{4n^2-1}$

$$
\begin{aligned}
&= \lim_{n\to\infty} \left(\sqrt{4n^2+n} - \sqrt{4n^2-1}\right) \cdot \frac{\left(\sqrt{4n^2+n} + \sqrt{4n^2-1}\right)}{\left(\sqrt{4n^2+n} + \sqrt{4n^2-1}\right)} \\
&= \lim_{n\to\infty} \frac{(4n^2+n) - (4n^2-1)}{\left(\sqrt{4n^2+n} + \sqrt{4n^2-1}\right)} \\
&= \lim_{n\to\infty} \frac{n+1}{\sqrt{4n^2+n} + \sqrt{4n^2-1}} \\
&= \lim_{n\to\infty} \frac{n+1}{n\left(\sqrt{4+\frac{1}{n}} + \sqrt{4-\frac{1}{n^2}}\right)} \\
&= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{\left(\sqrt{4+\frac{1}{n}} + \sqrt{4-\frac{1}{n^2}}\right)} \\
&= \frac{1}{4}.
\end{aligned}
$$

Exercise 1.4. • **WeBWorK**

1.1.3 Monotonic Sequences

Definition 1.5. *A sequence $\{a_n\}$ is said to be:*

- **increasing** *if $a_{n+1} \geq a_n$ for all n,*
- **decreasing** *if $a_{n+1} \leq a_n$ for all n.*

A sequence is said to be **monotonic** *if it is either increasing or decreasing.*

Theorem 1.6 (Monotone Convergence Theorem)**.** *If $\{a_n\}$ is either:*
increasing (i.e. $a_{n+1} \geq a_n$ for all n) and bounded above (i.e. There exists a number M such that $a_n \leq M$ for all n.), or
decreasing (i.e. $a_{n+1} \leq a_n$ for all n) and bounded below (i.e. There exists a number M such that $a_n \geq M$ for all n.), then $\{a_n\}$ converges.

Moreover,
if $\{a_n\}$ is increasing and $a_n \leq M$ for all n, then $\lim_{n\to\infty} a_n \leq M$.
If $\{a_n\}$ is decreasing and $a_n \geq M$ for all n, then $\lim_{n\to\infty} a_n \geq M$.

Example 1.7. *Let $\{a_n\}$ be a sequence of real numbers, which is defined by*

$$a_1 = 1 \qquad \textit{and} \qquad a_n = \frac{12a_{n-1}+12}{a_{n-1}+13} \textit{ for } n > 1.$$

1. *Prove that* $0 \leq a_n \leq 3$*. (Hint: Perhaps* **mathematical induction** *could be useful here.)*

2. *Prove that* $\{a_n\}$ *converges (i.e.* $\lim_{n\to\infty} a_n$ *exists), then find its limit.*

Solution. *1. First, we show that* $a_n \geq 0$ *for all* $n \in \mathbb{N}$*.*

Base Step : By definition, $a_1 = 1 \geq 0$*.*

Inductive Step : Suppose $a_n \geq 0$ *for some* $n \in \mathbb{N}$*. We want to show that* $a_{n+1} \geq 0$ *also.*

By the definition of the sequence, we have:

$$a_{n+1} = \frac{12a_n + 12}{a_n + 13}.$$

By the **induction hypothesis** *, i.e.* $a_n \geq 0$*, we have:*

$$a_n + 13 > 0 \quad \textit{and} \quad 12a_n + 12 \geq 0.$$

Hence, $a_{n+1} \geq 0$*.*

It now follows from the principle of mathematical induction that $a_n \geq 0$ *for all* $n \in \mathbb{N}$*.*

Similary, to show that $a_n \leq 3$*, we first observe that by definition* $a_1 = 1 \leq 3$*. Whenever* $a_n \leq 3$*, we have:*

$$\begin{aligned} 3 - a_{n+1} &= 3 - \frac{12a_n + 12}{a_n + 13} \\ &= \frac{3a_n + 39 - 12a_n - 12}{a_n + 13} \\ &= \frac{9(3 - a_n)}{a_n + 13} \geq 0, \end{aligned}$$

which implies that $a_{n+1} \leq 3$ *also. Hence, by mathematical induction we conclude that* $a_n \leq 3$ *for all* $n \in \mathbb{N}$*.*

---

2. Observe that for all $n \in \mathbb{N}$, we have:

$$\begin{aligned} a_{n+1} - a_n &= \frac{12a_n + 12}{a_n + 13} - a_n \\ &= \frac{12a_n + 12 - a_n^2 - 13a_n}{a_n + 13} \\ &= -\frac{a_n^2 + a_n - 12}{a_n + 13} \\ &= -\frac{(a_n - 3)(a_n + 4)}{a_n + 13} \\ &\geq 0, \end{aligned}$$

since $0 \leq a_n \leq 3$, as shown in Part 1.

This shows that $\{a_n\}$ is an increasing sequence bounded above by 3. Hence, the limit $L = \lim\limits_{n\to\infty} a_n$ exists, by the Monotone Convergence Theorem.

To find L, we take the limit as $n \to \infty$ of both sides of the equation:

$$a_n = \frac{12a_{n-1} + 12}{a_{n-1} + 13}.$$

That is:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{12a_{n-1} + 12}{a_{n-1} + 13},$$

which gives:

$$L = \frac{12L + 12}{L + 13},$$

since $\lim_{n\to\infty} a_{n-1} = \lim_{n\to\infty} a_n = L$.

The equation above implies that:

$$L^2 + L - 12 = 0,$$

which gives $L = 3$ or $L = -4$. Since the sequence $\{a_n\}$ is bounded below by 0, we may eliminate the case $L = -4$.

We conclude that:

$$\lim_{n\to\infty} a_n = 3.$$

1.1.4 Sandwich Theorem

Theorem 1.8 (Sandwich Theorem for Sequences)**.** *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be sequences such that:*

$$a_n \leq b_n \leq c_n$$

for all n *sufficiently large. If*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L,$$

then $\lim_{n\to\infty} b_n = L$ *also.*

MATH 1010DE Week 2

Functions

2.1 Sandwich Theorem - Continued

Example 2.1. *1. Find the following limit:* $\lim_{n\to\infty} \frac{\sin(2^n) + (-1)^n \cos(2^n)}{n^3}$.

---

2. - *Prove that* $\frac{2^n}{n!} \leq \frac{4}{n}$ *for all natural numbers* $n \geq 2$.
   - *Then, show that* $\lim_{n\to\infty} \frac{2^n}{n!} = 0$.

---

3. *Suppose* $0 < a < 1$. *Let* $b = \frac{1}{a} - 1$. *For* $n \geq 2$, *use the binomial theorem to show that*

$$\frac{1}{a^n} \geq \frac{n(n-1)}{2} b^2.$$

Then, show that:

$$\lim_{n\to\infty} na^n = 0.$$

Exercise 2.2. *Using the inequality:*

$$\frac{1}{\sqrt{n^2+n}} \leq \frac{1}{\sqrt{n^2+r}} \leq \frac{1}{\sqrt{n^2+1}}, \quad \textit{for } r = 1, 2, 3, \cdots, n,$$

prove that:

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}\right)=1.$$

Solution. *We have:*

$$\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}} \le \underbrace{\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+1}}+\cdots+\frac{1}{\sqrt{n^2+1}}}_{n \textit{ times}} = \frac{n}{\sqrt{n^2+1}},$$

and:

$$\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}} \ge \underbrace{\frac{1}{\sqrt{n^2+n}}+\frac{1}{\sqrt{n^2+n}}+\cdots+\frac{1}{\sqrt{n^2+n}}}_{n \textit{ times}} = \frac{n}{\sqrt{n^2+n}}.$$

Since:

$$\lim_{n\to\infty}\frac{n}{\sqrt{n^2+1}}=\lim_{n\to\infty}\frac{n}{n\sqrt{1+\frac{1}{n^2}}}=\lim_{n\to\infty}\frac{1}{1+\frac{1}{n^2}}=1,$$

and:

$$\lim_{n\to\infty}\frac{n}{\sqrt{n^2+n}}=\lim_{n\to\infty}\frac{n}{n\sqrt{1+\frac{1}{n}}}=\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}=1,$$

by the Sandwich Theorem we conclude that:

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}\right)=1.$$

2.2 Functions

Definition 2.3. *A function:*

$$f : A \longrightarrow B$$

is a rule of correspondence from one set A (called the **domain***) to another set B (called the* **codomain***).*

Under this rule of correspondence, each element $x \in A$ corresponds to exactly one element $f(x) \in B$, called the **value** *of f at x.*

In the context of this course, the domain A is usually some subset (intervals, union of intervals) of $\mathbb{R}$, while the codomain B is often presumed to be $\mathbb{R}$. Sometimes, the domain of a function is not explicitly given, and a function is simply defined by an expression in terms of an independent variable.

For example,

$$f(x) = \sqrt{\frac{x+1}{x-2}}$$

In this case, the domain of f is assumed to be the **natural domain** (or **maximal domain** , **domain of definition**), namely the largest subset of $\mathbb{R}$ on which the expression defining f is well-defined.

Example 2.4. *For the function:*

$$f(x) = \sqrt{\frac{x+1}{x-2}},$$

the natural domain is:

$$\begin{aligned}\text{Domain}(f) &= \left\{ x \in \mathbb{R} \;\middle|\; \frac{x+1}{x-2} \geq 0 \right\} \\ &= (-\infty, -1] \cup (2, \infty).\end{aligned}$$

2.2.1 Graphs of Functions

For $f : A \longrightarrow B$ where A, B are subsets of $\mathbb{R}$, it is often useful to consider the **graph** of f, namely the set of all points (x, y) in the xy-plane where $x \in A$ and $y = f(x)$. By definition, any function f takes on a unique value $f(x)$ for each x in its domain, hence the graph of f necessarily passes the so-called "**vertical line test**", namely, any vertical line which one draws in the xy-plane intersects the graph of f *at most once*.

The graph of a circle, for example, is not the graph of any function, since there are vertical lines which intersect the graph twice.

Exercise 2.5. *Graph the functions* $f(x) = \dfrac{x}{2}$ *and* $g(x) = \dfrac{4}{x} - 1$ *together, to identify values of* x *for which*

$$\frac{x}{2} > \frac{4}{x} - 1.$$

Confirm your answer by solving the inequality algebraically.

Solution. *The inequality holds if and only if:*

$$x \in (-4, 0) \cup (2, \infty)$$

2.2.2 Algebraic Operations on Functions

Definition 2.6. *Given two functions:*

$$f, g : A \longrightarrow \mathbb{R},$$

- *Their* **sum/difference** *is:*

$$f \pm g : A \longrightarrow \mathbb{R},$$
$$(f \pm g)(a) := f(a) \pm g(a), \quad \textit{for all } a \in A;$$

- *Their* **product** *is:*

$$fg : A \longrightarrow \mathbb{R},$$
$$fg(a) := f(a)g(a), \quad \textit{for all } a \in A;$$

- *The* **quotient function** $\dfrac{f}{g}$ *is:*

$$\frac{f}{g} : A' \longrightarrow \mathbb{R},$$
$$\frac{f}{g}(a) := \frac{f(a)}{g(a)}, \quad \textit{for all } a \in A',$$

where

$$A' = \{a \in A : g(a) \neq 0\}.$$

More generally, For:

$$f : A \longrightarrow \mathbb{R},$$
$$g : B \longrightarrow \mathbb{R},$$

we define $f \pm g$ and fg as follows:

$$f \pm g : A \cap B \longrightarrow \mathbb{R},$$
$$f \pm g(x) := f(x) \pm g(x), \quad x \in A \cap B.$$

$$fg : A \cap B \longrightarrow \mathbb{R},$$
$$fg(x) := f(x)g(x), \quad x \in A \cap B.$$

Similary, we define:

$$\frac{f}{g} : A \cap B' \longrightarrow \mathbb{R},$$
$$\frac{f}{g}(x) = \frac{f(x)}{g(x)}, \quad x \in A \cap B',$$

where $B' = \{b \in B : g(b) \neq 0\}$.

2.2.3 Composition of Functions

Given two functions:

$$f : A \longrightarrow B, \quad g : B \longrightarrow C,$$

the **composite function** $g \circ f$ is defined as follows:

$$g \circ f : A \longrightarrow C,$$
$$(g \circ f)(a) := g(f(a)), \quad \text{for all } a \in A.$$

More generally, the domain of $g \circ f$ is defined to be:

$$\text{Domain}(g \circ f) = \{a \in \text{Domain}(f) : f(a) \in \text{Domain}(g)\}.$$

2.2.4 Inverse of a Function

The **range** or **image** of a function $f : A \longrightarrow B$ is the set of all $b \in B$ such that $b = f(a)$ for some $a \in A$.

Notation.

$$\text{Image}(f) = \text{Range}(f) := \{b \in B \ : \ b = f(a) \textit{ for some } a \in A\}.$$

Note that the range of f is not necessarily equal to the codomain B.

Definition 2.7. *If* $\text{Range}(f) = B$*, we say that* f *is* **surjective** *or* **onto** .

Definition 2.8. *If* $f(a) \neq f(a')$ *for all* $a, a' \in \mathrm{Domain}(f)$ *such that* $a \neq a'$*, we say that* f *is* **injective** *or* **one-to-one** .

If $f : A \longrightarrow B$ is injective, then there exists an **inverse function**:

$$f^{-1} : \mathrm{Range}(f) \longrightarrow A$$

such that $f^{-1} \circ f$ is the **identity function** on A, and $f \circ f^{-1}$ is the identity function on $\mathrm{Range}(f)$, that is:

- $$f^{-1}(f(a)) = a, \qquad \text{for all } a \in A,$$
- $$f(f^{-1}(b)) = b, \qquad \text{for all } b \in \mathrm{Range}(f).$$

Example 2.9.

$$f : \mathbb{R} \longrightarrow \mathbb{R},$$
$$f(x) := x^2, \quad x \in \mathbb{R}.$$

is not injective, hence it has no inverse.

On the other hand,

$$f : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R},$$
$$f(x) := x^2, \quad x \in \mathbb{R}_{\geq 0};$$

is injective. It's range is $\mathrm{Range}(f) = \mathbb{R}_{\geq 0}$. Its inverse is:

$$f^{-1} : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$$

$$f^{-1}(y) = \sqrt{y}, \quad y \in \mathbb{R}_{\geq 0}.$$

Similarly,

$$g : \mathbb{R}_{\leq 0} \longrightarrow \mathbb{R},$$
$$g(x) := x^2, \quad x \in \mathbb{R}_{\leq 0};$$

is also injective, with $\mathrm{Range}(g) = \mathbb{R}_{\geq 0}$, and inverse:

$$g^{-1} : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\leq 0}$$

$$g^{-1}(y) = -\sqrt{y}, \quad y \in \mathbb{R}_{\geq 0}.$$

2.3 Piecewise Defined Functions

Example 2.10. •

$$f(x) = \begin{cases} -x+1 & \textit{if } -2 \leq x < 0 \\ 3x & \textit{if } 0 \leq x \leq 5 \end{cases}$$

- *The* **absolute value function**

$$|x| = \begin{cases} -x & \textit{if } x < 0 \\ x & \textit{if } x \geq 0 \end{cases}$$

Exercise 2.11. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by:*

$$f(x) = -3x + 4 - |x+1| - |x-1|$$

for any $x \in \mathbb{R}$.

1. *Express the 'explicit formula' of the function f as that of a piecewise defined function, with one 'piece' for each of $(-\infty, -1)$, $[-1, 1)$, $[1, +\infty)$.*

2. *Sketch the graph of the function f.*

3. *Is f an injective function on $\mathbb{R}$? Justify your answer.*

4. *What is the image of $\mathbb{R}$ under the function f?*

Solution.

1.
$$f(x) = \begin{cases} -x+4 & \textit{if} \quad x < -1 \\ -3x+2 & \textit{if} \quad -1 \leq x < 1 \\ -5x+4 & \textit{if} \quad x \geq 1 \end{cases}$$

2.

3. *f is strictly decreasing on $\mathbb{R}$. Hence, f is injective on $\mathbb{R}$.*

4. *The image of $\mathbb{R}$ under f is $\mathbb{R}$.*

2.4 WeBWorK

1. **WeBWorK**
2. **WeBWorK**
3. **WeBWorK**
4. **WeBWorK**
5. **WeBWorK**
6. **WeBWorK**

2.5 Even and Odd Functions

Definition 2.12. *Let f be a real-valued function defined on real numbers.*

- *It is said to be* **even** *if for any $x \in \mathrm{Domain}(f)$, $-x$ also lies in $\mathrm{Domain}(f)$ and:*
$$f(-x) = f(x).$$
- *It is said to be* **odd** *if for any $x \in \mathrm{Domain}(f)$, $-x$ also lies in $\mathrm{Domain}(f)$ and:*
$$f(-x) = -f(x).$$

Example 2.13.

1. *The polynomial $f(x) = x^4 + x^2 + 1$ is even, while the polynomial $g(x) = x^5 + x^3 + x$ is odd.*
2. *The function $f(x) = \cos x$ is even, while $f(x) = \sin x$ is odd.*
3. *The absolute value function is even.*

Fact 2.14.

1. *The sum of two even (resp. odd) functions is even (resp. odd).*
2. *The product of two even functions is even.*
3. *The product of two odd functions is also even.*
4. *The product of an even function with an odd function is odd.*
   For example, $f(x) = x\,|x|$ is odd.

MATH 1010DE Week 3

Functions, Limits, Sandwich Theorem

3.1 Limits of Functions on the Real Line

Let $f : A \longrightarrow \mathbb{R}$ be a function, where $A \subseteq \mathbb{R}$. Let a be a point on the real line such that f is defined on a neighborhood of a (though not necessarily at a itself).

Definition 3.1. *We say that the* **limit** *of f at a is L if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever x satisfies $0 < |x - a| < \delta$.*

If f has a limit L at a, we write:

$$\lim_{x \to a} f(x) = L.$$

Note that the limit may exist even if a does not lie in the domain of f.

Remark. *Intuitively, $\lim_{x \to a} f(x) = L$ means that the value $f(x)$ approaches L as x approaches a from either side, or that $f(x)$ is very near L whenever x is very near a. Obviously, the term "near" is somewhat vague, and it is precisely because of this vagueness that mathematicians feel the need to define limits rigorously using the "δ-ε" language.*

Example 3.2. *Consider $f(x) = \dfrac{x^2 - 4}{x + 2}$. Note that the function f is not defined at -2.*

Observe that for x near -2, for example, $x = -2.001$, or $x = -1.9999$, we have:

$$f(-2.001) = -4.001,$$

$$f(-1.9999) = -3.9999,$$

which are close to -4.

Moreover, as x "approaches" -2 ($x = -2.001, -2.0001, -2.00001, \ldots$), we have $f(x) = -4.001, -4.0001, -4.00001$. So, it appears $f(x)$ approaches -4 as x approaches -2. This suggests that the limit of $f(x)$ at $x = -2$ is:

$$\lim_{x \to -2} f(x) = -4.$$

This turns out to be true, and is not surprising, since we can rewrite $f(x)$ as follows:

$$f(x) = \begin{cases} \dfrac{(x+2)(x-2)}{x+2}, & \textit{if } x \neq -2; \\ \textit{undefined}, & \textit{if } x = -2. \end{cases}$$
$$= \begin{cases} x-2, & \textit{if } x \neq -2; \\ \textit{undefined}, & \textit{if } x = -2. \end{cases}$$

Hence, all along we have really been asking what $x - 2$ tends to as x tends to -2.

Definition 3.3. *Let $f : A \longrightarrow \mathbb{R}$ be a function, where $A \subseteq \mathbb{R}$ is unbounded towards $+\infty$ and/or $-\infty$. We say that the* **limit** *of f at ∞ (resp. $-\infty$) is L if for all $\varepsilon > 0$, there exists a $c \in \mathbb{R}$ such that $|f(x) - L| < \varepsilon$ whenever $x > c$ (resp. $x < c$).*

If f has a limit L at ∞ (resp $-\infty$), we write:

$$\lim_{x \to \infty} f(x) = L \quad \left(\text{resp. } \lim_{x \to -\infty} f(x) = L\right)$$

3.1.1 Some Useful Identities

In the following idenities, the symbol a can be either a real number or $\pm\infty$.

1. For any constant $c \in \mathbb{R}$, we have $\lim\limits_{x \to a} c = c$.
2. $\lim\limits_{x \to a} x = a$.
3. If $\lim_{x \to a} f(x) = L$, and $\lim_{x \to a} g(x) = M$, then:
   - $\lim_{x \to a}(f \pm g)(x) = L \pm M$.
   - $\lim_{x \to a} fg(x) = LM$.
   - $$\lim_{x \to a} \frac{f}{g}(x) = \frac{L}{M}$$

     provided that $M \neq 0$.

4. If $\lim_{x\to a} f(x) = L$, then:

$$\lim_{x\to a}(f(x))^n = L^n \quad \text{for all } n \in \mathbb{N} = \{1, 2, 3, \ldots\},$$

and

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{L} \quad \text{for all odd positive integers } n.$$

In particular, for all positive integer n, we have:

$$\lim_{x\to a} x^n = a^n.$$

5. If $\lim_{x\to a} f(x) = L > 0$, then $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{L}$ for all $n \in \mathbb{N}$.

Example 3.4. *Compute the following limits, if they exist:*

- $\lim_{x\to -1} \dfrac{x^2 - 1}{x^2 - 5x - 6}$

- $\lim_{x\to 4} \dfrac{2 - \sqrt{x}}{16 - x^2}$

3.2 WeBWorK

1. **WeBWorK**
2. **WeBWorK**
3. **WeBWorK**
4. **WeBWorK**
5. **WeBWorK**

3.3 One-Sided Limits

- We write $\lim_{x\to a^+} f(x) = L$ if $f(x)$ approaches L as x approaches a *from the right* . We call this L the **right limit** of f at a.
- Similarly, we write $\lim_{x\to a^-} f(x) = L$ if $f(x)$ approaches L as x approaches a *from the left* . We call this L the **left limit** of f at a.

The limit $\lim_{x \to a} f(x)$ is sometimes called the **double-sided limit** of f at a. It exists if and only if both one-sided limits exist and are equal to each other. In which case, we have:

$$\lim_{x \to a} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x).$$

Exercise 3.5. *Define*

$$f(x) = \begin{cases} x - 1 & \textit{if } 1 \leq x \leq 2, \\ 2x + 3 & \textit{if } 2 < x \leq 4, \\ x^2 & \textit{otherwise.} \end{cases}$$

Compute $\lim_{x \to 2^+} f(x)$ *and* $\lim_{x \to 2^-} f(x)$. *Then, find* $\lim_{x \to 2} f(x)$, *if it exists.*

Answers.

1.

$$\lim_{x \to 2^+} f(x) = 7$$
$$\lim_{x \to 2^-} f(x) = 1$$

2. Since $\lim_{x \to 2^+} f(x) \neq \lim_{x \to 2^-} f(x)$, *the double-sided limit* $\lim_{x \to 2} f(x)$ *does not exist.*

3.4 WeBWorK

1. **WeBWorK**
2. **WeBWorK**
3. **WeBWorK**
4. **WeBWorK**
5. **WeBWorK**

3.5 Sandwich Theorem for Functions on the Real Line

Theorem 3.6. *Let* $a \in \mathbb{R}$, A *an open neighborhood of* a *which does not necessarily contain* a *itself. Let* $f, g, h : A \longrightarrow \mathbb{R}$ *be functions such that:*

$$g(x) \leq f(x) \leq h(x) \quad \textit{for all } x \in A,$$

and

$$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L.$$

Then, $\lim_{x\to a} f(x) = L$*.*

Similary,

Theorem 3.7. *If f, g, h are functions on $\mathbb{R}$ such that:*

$$g(x) \le f(x) \le h(x)$$

for all x sufficiently large, and

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} h(x) = L,$$

then $\lim_{x\to\infty} f(x) = L$*.*

Exercise 3.8. *Find the following limits, if they exist:*

- $\lim_{x\to\infty} \dfrac{\sin x}{x}$

- $\lim_{x\to\infty} \dfrac{x+\sin x}{x-\sin x}$

Theorem 3.9.

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Corollary 3.10.

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}\ .$$

Proof.

$$\begin{aligned}
\lim_{x\to 0} \frac{1-\cos x}{x^2} &= \lim_{x\to 0} \frac{1-\cos x}{x^2} \cdot \left(\frac{1+\cos x}{1+\cos x}\right) \\
&= \lim_{x\to 0} \frac{1-\cos^2 x}{x^2\,(1+\cos x)} \\
&= \lim_{x\to 0} \frac{\sin^2 x}{x^2\,(1+\cos x)} \\
&= \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2 \frac{1}{1+\cos x} \\
&= 1^2 \cdot \frac{1}{1+1} = \frac{1}{2}
\end{aligned}$$

□

Corollary 3.11.

$$\lim_{x\to 0} \frac{1-\cos x}{x} = 0\ .$$

Exercise 3.12. *Find the following limits, if they exist:*

- $\lim_{x\to 0} \frac{\sin(5x)}{\tan(3x)}$

- $\lim_{x\to 0} \frac{x^3 \cos\left(\frac{1}{x}\right)}{\tan x}$

3.6 WeBWorK

1. **WeBWorK**
2. **WeBWorK**
3. **WeBWorK**
4. **WeBWorK**
5. **WeBWorK**
6. **WeBWorK**

MATH 1010DE Week 4

Limits, Continuity

4.1 More Limit Identities

Example 4.1. *Find:*

- $\lim_{x \to 0^+} \sin\left(\frac{1}{x}\right)$
- $\lim_{x \to 0^+} x \sin\left(\frac{1}{x}\right)$

Definition 4.2. *For each $x \in \mathbb{R}$, we let:*

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

It is known that:

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n.$$

Theorem 4.3.

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e$$

Corollary 4.4.

$$\lim_{x \to \infty} \left(1 - \frac{1}{x}\right)^x = \lim_{x \to 0} (1 - x)^{\frac{1}{x}} = \frac{1}{e}$$

For all $a \in \mathbb{R}$,

$$\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x = e^a$$

Exercise 4.5. *Find:*

$$\lim_{x\to\infty}\left(\frac{x+1}{x-1}\right)^x$$

Theorem 4.6. *For all $n \in \{1, 2, 3, \ldots\}$, we have:*

$$\lim_{x\to\infty}\frac{x^n}{e^x}=0.$$

Corollary 4.7. *For all $n \in \{1, 2, 3, \ldots\}$, and $b > 1$, we have:*

$$\lim_{x\to\infty}\frac{x^n}{b^x}=0.$$

Fact 4.8.

$$\lim_{x\to 0}\frac{\ln(1+x)}{x}=1.$$

From this may be further deduced that:

$$\lim_{t\to 0}\frac{e^t-1}{t}=1,$$

by applying a change of variable:

$$x=e^t-1.$$

4.2 Continuity

Definition 4.9. *A function $f : A \longrightarrow \mathbb{R}$ is said to be* **continuous** *at $c \in A$ if:*

$$\lim_{x\to c} f(x)=f(c).$$

A function is said to be **continuous** *if it is continuous at every point in its domain.*

Should c be an endpoint in the domain of f, the continuity of f at c is defined in terms of a one-sided limit. That is, right limit if c is a left endpoint, and left limit if c is a right endpoint. Hence, the function:

$$f(x)=\sqrt{x}$$

is continuous at $x = 0$, since $\mathrm{Domain}(f) = [0, \infty)$, and:

$$\lim_{x\to 0^+} f(x)= \ 0 \ = f(0).$$

The following "elementary functions" are continuous at every element in their domains:

$$f(x) = x, \frac{1}{x}, \sin x, \cos x, \tan x, e^x, \ln x, \arcsin x, \arccos x, \arctan x$$

Due to the laws of sum/difference/product/quotient for limits, the sum/difference/product/quotient of continuous functions is also continous.

In particular, polynomials and rational functions are all continuous on their domains.

Theorem 4.10. *For functions g, f with the property that $\lim_{x\to a} g(x)$ exists and f is continuous at $\lim_{x\to a} g(x)$, we have:*

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right).$$

Example 4.11. *It follows from this theorem that:*

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$$

It also follows from the previous theorem that any composite of continuous functions is continuous.

Example 4.12. *The following functions are all continuous, since they are the sums, differences, products, quotients, or composites of other continuous functions:*

$$f(x) = \frac{e^{\cos\left(\frac{1}{x}\right)}}{x^7 - 9x^2 + 23}$$
$$g(x) = \frac{1}{\arctan x} - \sqrt[3]{\log_5\left(2^x + 1\right)}$$
$$h(x) = \sin\left(x^{-3} + \left(\cos\left(e^{x^2} + 1\right)\right)\right)$$

Example 4.13. *The following functions are continuous at every point on the real line:*

- $$g(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0; \\ 1, & x = 0; \end{cases}$$

- $$f(x) = \begin{cases} x^2 \cos\left(\dfrac{1}{e^x - 1}\right), & x \neq 0; \\ 0, & x = 0; \end{cases}$$

Exercise 4.14. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function that satisfies:*

- $f(x+y) = f(x)f(y)$ *for all* $x, y \in \mathbb{R}$*;*
- $f(x)$ *is continuous at* $x = 0$ *and* $f(0) \neq 0$.

---

1. *Show that* $f(0) = 1$.
2. *Show that* $f(x)$ *is continuous on* $\mathbb{R}$.

4.2.1 WeBWorK

1. **WeBWorK**
2. **WeBWorK**
3. **WeBWorK**
4. **WeBWorK**

4.2.2 Further Properties of Continuous Functions

Theorem 4.15 (Intermediate Value Theorem IVT)**.** *If $f : [a,b] \longrightarrow \mathbb{R}$ is continuous, then f attains every value between $f(a)$ and $f(b)$. In other words, for any $y \in \mathbb{R}$ between the values of $f(a)$ and $f(b)$, there exists $c \in [a,b]$ such that $f(c) = y$.*

Exercise 4.16.

- *Show that* $f(x) = x^5 + x^2 - 10 = 0$ *has a real root between* $x = 1$ *and* $x = 2$.
- *Show that the range of* $f(x) = e^x - \sqrt{x}$ *contains* $[1, \infty)$.

Theorem 4.17 (Extreme Value Theorem)**.** *If f is a continuous function defined on a closed interval $[a,b]$, then it attains both a maximum value and a minimum value on $[a,b]$.*

MATH 1010DE Week 5

Differentiation

5.1 Derivatives

Definition 5.1. *We say that a function f is* **differentiable** *at c if the limit:*

$$f'(c) := \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists. The limit $f'(c)$, if it exists, is called the **derivative** *of f at c.*

Interactive Example

We say that a function f is **differentiable** if it is differentiable at every point in its domain.

Exercise 5.2. *Let $f(x) = |x|$. Is f differentiable at $x = 0$? If so, find $f'(0)$.*

Theorem 5.3. *If a function f is differentiable at c, then it is also continuous at c. (The converse is false in general.)*

Example 5.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by*

$$f(x) = \begin{cases} x^3 & \text{if } x \leq 1; \\ ax + b & \text{if } x > 1. \end{cases}$$

Suppose $f(x)$ differentiable at $x = 1$, find the values of a and b.

5.2 WeBWorK

1. **WeBWorK**

2. **WeBWorK**

3. **WeBWorK**

4. **WeBWorK**

5. **WeBWorK**

6. **WeBWorK**

7. **WeBWorK**

8. **WeBWorK**

5.3 Tangent Line

If the derivative $f'(c)$ exists, then there exists a tangent line to the graph $y = f(x)$ of f at $(c, f(c))$. Moreover, the slope of the tangent line is $f'(c)$, and the tangent line is the graph of the equation:

$$y = f'(c)(x - c) + f(c).$$

Given $f : A \longrightarrow \mathbb{R}$, the correspondence $x \mapsto f'(x)$ defines the **derivative function** $f' : A' \longrightarrow \mathbb{R}$, where A' is the set of all points $c \in A$ at which f is differentiable.

5.4 Some Common Derivative Identities

$f(x)$	$f'(x)$
constant	0
$ax+b \quad (a, b \in \mathbb{R})$	a
$x^n \quad (n \in \mathbb{Z},\ n \neq 0, 1)$	nx^{n-1}
$x^r \quad (r \in \mathbb{R},\ x > 0)$	rx^{r-1}
e^x	e^x
$a^x \quad (a > 0)$	$(\ln a)a^x$
$\ln \lvert x \rvert$	$\frac{1}{x}$
$\log_a x \quad (a \neq 1,\ a > 0)$	$\frac{1}{(\ln a)x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\csc^2 x$
$\csc x$	$-\csc x \cot x$
$\arctan x$	$\frac{1}{x^2+1}$
$\arcsin x \quad (-1 < x < 1)$	$\frac{1}{\sqrt{1-x^2}}$

5.5 Leibniz Notation

If f is defined in terms of an independent variable x, we often denote $f'(x)$ by $\frac{df}{dx}$. Under this notation, for a given $c \in \mathbb{R}$ the value $f'(c)$ is denoted by:

$$\left.\frac{df}{dx}\right|_{x=c}$$

5.6 Rules of Differentiation

Let f, g be functions differentiable at $c \in \mathbb{R}$. Then:

Sum/Difference Rule

$f \pm g$ is differentiable at c, with:

$$(f \pm g)'(c) = f'(c) \pm g'(c).$$

Proof.

$$\begin{aligned}(f+g)'(c) &= \lim_{h\to 0}\frac{(f+g)(c+h)-(f+g)(c)}{h}\\ &= \lim_{h\to 0}\frac{f(c+h)+g(c+h)-f(c)-g(c)}{h}\\ &= \lim_{h\to 0}\left[\frac{f(c+h)-f(c)}{h}+\frac{g(c+h)-g(c)}{h}\right]. \qquad (*)\end{aligned}$$

Since by assumption both $f'(c)=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$ and $g'(c)=\lim_{h\to 0}\frac{g(c+h)-g(c)}{h}$ exist, by the sum rule for limits the expression $(*)$ is equal to:

$$f'(c)+g'(c).$$

Exercise. Show that $(f-g)'(c)=f'(c)-g'(c)$. □

Product Rule

fg is differentiable at c, with:

$$(fg)'(c)=f'(c)g(c)+f(c)g'(c).$$

Quotient Rule

f/g is differentiable at c provided that $g(c)\neq 0$, in which case we have:

$$\left(\frac{f}{g}\right)'(c)=\frac{g(c)f'(c)-f(c)g'(c)}{[g(c)]^2}\quad .$$

5.7 Chain Rule

Theorem 5.5. *Suppose f is differentiable at c and g is differentiable at $f(c)$, then $g\circ f$ is differentiable at c, with:*

$$(g\circ f)'(c)=g'(f(c))f'(c).$$

In the Leibniz notation, the chain rule says that if f is a differentiable function of u and u is a differentiable function of x, then:

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} \, ,$$

$$\left.\frac{df}{dx}\right|_{x=c} = \left.\frac{df}{du}\right|_{u=u(c)} \left.\frac{du}{dx}\right|_{x=c}$$

Exercise 5.6. *Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by*

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x^2}\right) & \textit{if } x \neq 0; \\ 0 & \textit{if } x = 0. \end{cases}$$

Find f'.

5.8 WeBWorK

1. **WeBWorK**
2. **WeBWorK**
3. **WeBWorK**
4. **WeBWorK**
5. **WeBWorK**
6. **WeBWorK**
7. **WeBWorK**
8. **WeBWorK**
9. **WeBWorK**
10. **WeBWorK**
11. **WeBWorK**
12. **WeBWorK**

MATH 1010DE Week 6

Implicit Differentiation, Higher Order Derivatives

6.1 Implicit Differentiation

Example 6.1. *For $x > 0$,*

$$\frac{d}{dx} \ln x = \frac{1}{x} .$$

Proof. Consider the equation:

$$e^{\ln x} = x$$

Differentiating both sides with respect to x, and applying the Chain Rule, we have:

$$\begin{aligned} \frac{d}{dx} e^{\ln x} &= \frac{d}{dx} x \\ \underbrace{e^{\ln x}}_{=x} \frac{d}{dx} \ln x &= 1 \end{aligned}$$

Hence, $\dfrac{d}{dx} \ln x = \dfrac{1}{x}$. □

Example 6.2. *Find $\dfrac{d}{dx} (x^x)$, where $x > 0$.*

For any $x > 0$, we have $x = e^{\ln x}$. Hence,

$$x^x = \left(e^{\ln x}\right)^x = e^{x \ln x}.$$

So,

$$\begin{aligned}\frac{d}{dx}\left(x^x\right) &= \frac{d}{dx}e^{x\ln x}\\ &= e^{x\ln x}\frac{d}{dx}(x\ln x) \quad (\textit{ by the Chain Rule.})\\ &= e^{x\ln x}\left(x\cdot\frac{1}{x}+\ln x\right) \quad (\textit{ by the Product Rule.})\\ &= e^{x\ln x}(1+\ln x) \quad (\textit{ since } x>0.)\\ &= (1+\ln x)x^x.\end{aligned}$$

Exercise 6.3. *Consider the curve $C : y^4 - y\cos(x) - x^4 = 0$.*

1. *Find $\frac{dy}{dx}$. Express your answer in terms of x, y only.*
2. *Let $P = \left(\frac{\pi}{2}, -\frac{\pi}{2}\right)$.*
   - *Verify that the point P lies on the curve C.*
   - *Find the equation of the tangent line to the curve C at the point P.*

Solution. *First, we differentiate both sides of the equation $y^4 - y\cos(x) - x^4 = 0$ with respect to x:*

$$\frac{d}{dx}(y^4 - y\cos(x) - x^4) = \frac{d}{dx}0 \tag{6.1}$$

By the chain rule, we have:

$$\frac{d}{dx}y^4 = \frac{d\left(y^4\right)}{dy}\frac{dy}{dx} = 4y^3\frac{dy}{dx}.$$

Hence, equation (6.1) gives:

$$4y^3\frac{dy}{dx} - \left(y(-\sin(x)) + \frac{dy}{dx}\cdot\cos(x)\right) - 4x^3 = 0.$$

Grouping all the terms involving $\frac{dy}{dx}$ together, we have:

$$\left(4y^3 - \cos x\right)\frac{dy}{dx} = 4x^3 - y\sin x$$

Hence,

$$\frac{dy}{dx} = \frac{4x^3 - y\sin x}{4y^3 - \cos x}$$

The tangent line to the curve C at the point $(\pi/2, -\pi/2)$ is equal to:

$$\left.\frac{dy}{dx}\right|_{(\pi/2,-\pi/2)} = \frac{4(\pi/2)^3 + \pi/2}{-4(\pi/2)^3}$$

Hence, the equation of the tangent line is:

$$y = \left(\frac{4(\pi/2)^3 + \pi/2}{-4(\pi/2)^3}\right)(x - \pi/2) - \pi/2$$

Theorem 6.4. *Let f be an injective function differentiable at $x = c$. If $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$, with:*

$$\left(f^{-1}\right)'(f(c)) = \frac{1}{f'(c)}\ .$$

Equivalently, for any $y \in \operatorname{Range}(f)$, if f is differentiable at $x = f^{-1}(y)$, and $f'(f^{-1}(y)) \neq 0$, then:

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(f^{-1}(y))}\ .$$

Example 6.5. *Consider the injective function:*

$$f : [-\pi/2, \pi/2] \longrightarrow \mathbb{R},$$
$$f(x) = \sin x, \quad x \in [-\pi/2, \pi/2].$$

The inverse of f is:

$$f^{-1} = \arcsin : [-1, 1] \longrightarrow [-\pi/2, \pi/2].$$

Consider any $y \in (-1, 1)$. We have $y = f(x) = \sin(x)$ for a unique $x = \arcsin y$ in $(-\pi/2, \pi/2)$. Since $x \in (-\pi/2, \pi/2)$, we have $f'(x) = \cos(x) \neq 0$.

Hence, by Theorem 6.4, $(f^{-1})'(y)$ exists, with:

$$\left(f^{-1}\right)'(y) = \left(f^{-1}\right)'(f(x)) = \frac{1}{f'(x)} = \frac{1}{\cos x}\ .$$

By the Pythagorean Theorem, we know that:

$$\cos x = \pm\sqrt{1 - \sin^2 x}\ .$$

Moreover, since $x \in (-\pi/2, \pi/2)$, we have $\cos x > 0$, so:

$$\cos x = +\sqrt{1 - \sin^2 x} = \sqrt{1 - \sin^2(\arcsin(y))} = \sqrt{1 - y^2}.$$

In conclusion, for $y \in (-1, 1)$, we have:

$$\arcsin' y = \left(f^{-1}\right)'(y) = \frac{1}{\sqrt{1 - y^2}}\ .$$

Example 6.6. *Similary, we can find the derivative of* $\arccos$ *as follows:*

The function $\arccos$ *is the inverse function* g^{-1} *of the following injective function:*

$$g(x) = \cos x, \quad x \in [0, \pi].$$

For any $y \in (-1, 1)$*, we have* $g^{-1}(y) \in (0, \pi)$*, so* $g'(g^{-1}(y)) = -\sin(\arccos(y)) \neq 0$*. Hence, by Theorem 6.4, the function* g^{-1} *is differentiable at* $y \in (-1, 1)$*, with:*

$$(g^{-1})'(y) = \frac{1}{g'(g^{-1}(y))} = \frac{1}{-\sin(\arccos(y))}.$$

By the Pythagorean Theorem, $\sin x = \pm\sqrt{1 - \cos^2(x)}$*. Since* $\arccos(y) \in (0, \pi)$ *for* $y \in (-1, 1)$*, we have:*

$$\sin(\arccos(y)) = +\sqrt{1 - \cos^2(\arccos(y))} = \sqrt{1 - y^2}.$$

Hence,

$$\arccos' y = (g^{-1})'(y) = -\frac{1}{\sqrt{1 - y^2}}.$$

6.2 WeBWorK

1. **WeBWorK**
2. **WeBWorK**
3. **WeBWorK**
4. **WeBWorK**
5. **WeBWorK**
6. **WeBWorK**

6.3 Higher Order Derivatives

Let f be a function.

Its derivative f' is often called the **first derivative** of f.

The derivative of f', denoted by f'', is called the **second derivative** of f.

If $f''(c)$ exists, we say that f is **twice differentiable** at c.

For $n \in \mathbb{N}$, the n-th derivative of f, denoted by $f^{(n)}$ is defined as the derivative of the $(n-1)$-st derivative of f.

If $f^{(n)}(c)$ exists, we say that f is n times differentiable at c.

We sometimes consider f to be the "zero"-th derivative of itself, i.e. $f^{(0)} := f$.

In the Leibniz notation, we have:

$$f^{(n)}(x) = \underbrace{\frac{d}{dx}\frac{d}{dx}\cdots\frac{d}{dx}}_{n \text{ times}} f,$$

which is customarily written as:

$$\frac{d^n f}{dx^n} \quad .$$

Example 6.7. *Consider the curve:*

$$x^2 + y^2 = 1$$

Find $\frac{d^2y}{dx^2}$.

Solution. *Applying implicit differentiation, we have:*

$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}1$$

$$2x + 2y\frac{dy}{dx} = 0 \tag{6.2}$$

This shows that:

$$\frac{dy}{dx} = -\frac{x}{y}.$$

Applying implicity differentiation to equation (6.2)*, we have:*

$$\frac{d}{dx}\left(2x + 2y\frac{dy}{dx}\right) = \frac{d}{dx}0$$

$$2 + 2\left(y\frac{d^2y}{dx^2} + \frac{dy}{dx}\frac{dy}{dx}\right) = 0$$

It follows that:

$$\begin{aligned} y\frac{d^2y}{dx^2} &= -1 - \left(\frac{dy}{dx}\right)^2 \\ &= -1 - \frac{x^2}{y^2} \\ &= -\left(\frac{x^2+y^2}{y^2}\right) = -\left(\frac{1}{y^2}\right) \end{aligned}$$

Hence,

$$\frac{d^2y}{dx^2} = -\left(\frac{1}{y^3}\right)$$

Example 6.8. *Let:*

$$f(x) = \begin{cases} x^4 \sin\left(\frac{1}{x}\right) & \textit{if } x \neq 0; \\ 0 & \textit{if } x = 0. \end{cases}$$

Find $f''(0)$, if it exists.

Solution. *For $x \neq 0$, we have:*

$$\begin{aligned} f'(x) &= \frac{d}{dx}x^4\sin(1/x) \\ &= 4x^3\sin(1/x) + x^4\cos(1/x)\cdot(-x^{-2}) \\ &= 4x^3\sin(1/x) - x^2\cos(1/x) \\ &= x^2(4x\sin(1/x) - \cos(1/x)) \end{aligned}$$

By the limit definition of the derivative, we have:

$$\begin{aligned} f'(0) &= \lim_{h\to 0}\frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0}\frac{h^4\sin(1/h) - 0}{h} \\ &= \lim_{h\to 0} h^3\sin(1/h) = 0 \quad (\textit{by Sandwich Theorem}) \end{aligned}$$

Hence,

$$f'(x) = \begin{cases} x^2(4x\sin(1/x) - \cos(1/x)), & x \neq 0; \\ 0, & x = 0. \end{cases}$$

By definition:

$$f''(0) = (f')'(0) = \lim_{h\to 0} \frac{f'(0+h) - f'(0)}{h}.$$

Hence,

$$\begin{aligned} f''(0) &= \lim_{h\to 0} \frac{h^2(4h\sin(1/h) - \cos(1/h)) - 0}{h} \\ &= \lim_{h\to 0} h(4h\sin(1/h) - \cos(1/h)) \\ &= 0 \;\; (\textit{again by Sandwich Theorem}). \end{aligned}$$

Theorem 6.9 (General Leibniz Rule)**.** *Let $n \in \mathbb{N}$. Given any functions f, g which are n times differentiable at c, their product fg is also n times differentiable at c, with:*

$$(fg)^{(n)}(c) = \sum_{k=0}^{n} C_k^n f^{(k)}(c) g^{(n-k)}(c)$$

Notice that when $n = 1$ this rule is simply the product rule we have introduced before.

Example 6.10. *Consider $h(x) = x^2\sin(x)$. Then, $h = fg$, where $f(x) = x^2$ and $g(x) = \sin(x)$.*

We have:

$$f'(x) = 2x, \quad f''(x) = 2, \quad f^{(3)}(x) = 0.$$
$$g'(x) = \cos(x), \quad g''(x) = -\sin x, \quad g^{(3)}(x) = -\cos(x).$$

Hence, by the General Leibniz Rule, the first, second and third derivatives of h may be computed as follows:

$$\begin{aligned} h'(x) &= fg'(x) + f'g(x) \\ &= x^2\cos(x) + 2x\sin(x) \end{aligned}$$

$$\begin{aligned} h''(x) &= fg''(x) + 2f'g'(x) + f''g(x) \\ &= x^2(-\sin(x)) + 2(2x)\cos(x) + 2\sin(x) \end{aligned}$$

$$\begin{aligned} h^{(3)}(x) &= fg^{(3)}(x) + 3f'g''(x) + 3f''g'(x) + f^{(3)}g(x) \\ &= x^2(-\cos(x)) + 3(2x)(-\sin(x)) + 3(2)\cos(x) + 0\cdot\sin(x) \\ &= -x^2\cos(x) - 6x\sin(x) + 6\cos(x) \end{aligned}$$

6.4 WeBWorK

1. **WeBWorK**
2. **WeBWorK**
3. **WeBWorK**
4. **WeBWorK**
5. **WeBWorK**
6. **WeBWorK**
7. **WeBWorK**

MATH 1010DE Week 7

Mean Value Theorem

Theorem 7.1 (Extreme Value Theorem)**.** *If f is a continuous function defined on a closed interval $[a, b]$, then it attains both a maximum value and a minimum value on $[a, b]$.*

7.1 The Mean Value Theorem

Theorem 7.2 (Rolle's Theorem)**.** *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function which is continuous on $[a, b]$ and differentiable on (a, b) (i.e. $f'(x)$ exists for all $x \in (a, b)$). If $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.*

Interactive Example

Proof. Sketch of Proof. First, it follows from the Extreme Value Theorem that f has an absolute maximum or minimum at a point c in (a, b). It may then be shown that:

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = 0,$$

using that fact that if $f(c)$ is an absolute extremum, then $\frac{f(c+h) - f(c)}{h}$ is both ≤ 0 and ≥ 0. □

Theorem 7.3 (Mean Value Theorem MVT)**.** *(Also known as* **Lagrange's Mean Value Theorem***)*

If a function $f : [a, b] \longrightarrow \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Interactive Example

Proof. Let f be a function which satisfies the conditions of the theorem. Define a function $g : [a, b] \longrightarrow \mathbb{R}$ as follows:

$$g(x) = f(x) - \left[\left(\frac{f(b) - f(a)}{b - a} \right) (x - a) + f(a) \right], \quad x \in [a, b].$$

(Intuitively, g is obtained from f by subtracting from f the line segment joining $(a, f(a))$ and $(b, f(b))$.) Observe that:

$$g(a) = g(b) = 0,$$

so the function g satisfies the conditions of Rolle's Theorem. Hence, there exists $c \in (a, b)$ such that:

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a},$$

which implies that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. □

7.2 WeBWorK

1. **WeBWorK**
2. **WeBWorK**
3. **WeBWorK**

7.3 Applications of the Mean Value Theorem

Theorem 7.4. *Let f be a differentiable function on an open interval (a, b). If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on (a, b).*

Proof. Exercise. For any $x_1, x_2 \in (a, b)$, show that the difference $f(x_2) - f(x_1)$ is equal to 0. □

Theorem 7.5. *Let f be a differentiable function on an open interval (a, b). If $f'(x) > 0$ (resp. $f'(x) < 0$) for all $x \in (a, b)$, then f is* **strictly increasing** *(resp.* **strictly decreasing***) on (a, b).*

Remark: If f is moreover continuous on $[a, b]$, then f is strictly increasing (resp. strictly decreasing) on $[a, b]$ if f' is positive (resp. negative) on (a, b).

Proof. We will prove the case $f'(x) > 0$.

Suppose $f'(x) > 0$ for all $x \in (a, b)$. Given any $x_1, x_2 \in (a, b)$, such that $x_1 < x_2$, by the MVT there exists $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

By the condition $f'(x) > 0$ for all $x \in (a, b)$, we have $f'(c) > 0$. Also, $x_2 - x_1 > 0$. Hence:

$$f(x_2) - f(x_1) = f'(c) \cdot (x_2 - x_1) > 0.$$

This shows that f is strictly increasing on (a, b). □

Example 7.6. *Find the intervals where the function $f(x) = x^3 + 6x^2 - 15x + 7$ is increasing/decreasing.*

Solution. *We apply Theorem 7.5.*

First, we find:

$$f'(x) = 3x^2 + 12x - 15$$

Observe that f' is defined and continuous everywhere. Hence, the intervals where f' is positive/negative are separated by points c where $f'(c) = 0$. (Such points are called **stationary points** *of f).*

Setting:

$$f'(c) = 3c^2 + 12c - 15 = 3(c^2 + 4c - 5) = 3(c + 5)(c - 1) = 0,$$

we see that the points where f' possibly changes sign are:

$$c = -5, 1$$

Consider now the **sign chart** *:*

f:	$\nearrow$		$\searrow$		$\nearrow$
$f'(x)$:	$+$	0	$-$	0	$+$
x:	$(-\infty, -5)$	-5	$(-5, 1)$	1	$(1, \infty)$

It now follows from Theorem 7.5 and the continuity of f that:

- *f is strictly increasing on the intervals $(-\infty, -5]$ and $[1, \infty)$.*
- *f is strictly decreasing on the interval $[-5, 1]$.*

Example 7.7. *Let:*

$$f(x) = \begin{cases} (x+1)^2, & x < 0; \\ x+1, & x \geq 0. \end{cases}$$

Find the intervals where the function f is increasing/decreasing.

Solution. *We carry out the same steps as in the previous example. We leave it as an exercise to show that:*

$$f'(x) = \begin{cases} 2x+2, & x < 0; \\ \textit{undefined}, & x = 0; \\ 1, & x > 0. \end{cases}$$

Note that f' is not defined everywhere. In this case, the points where f' possibly changes sign are points c where:

$$f'(c) = 0 \quad \textit{or} \quad f'(c) \textit{ is undefined.}$$

Such points are called the **critical points** *of f. (Note that the set of stationary points is a subset of critical points).*

Constructing a sign chart as in the previous example, we have:

f:	$\searrow$		$\nearrow$		$\nearrow$
$f'(x)$:	$-$	0	$+$	*undefined*	$+$
x:	$(-\infty, -1)$	-1	$(-1, 0)$	0	$(0, \infty)$

Hence, by Theorem 7.5 , f is strictly decreasing on:

$$(-\infty, -1],$$

and strictly increasing on both $[-1, 0]$ and $[0, \infty)$. Since f is continuous at $x = 0$, we conclude that f is strictly increasing on:

$$[-1, \infty).$$

Exercise 7.8. *Use the mean value theorem to prove that for $x > 0$,*

$$\frac{x}{1+x} < \ln(1+x) < x.$$

Hence, deduce that for $x > 0$,

$$\frac{1}{1+x} < \ln\left(1 + \frac{1}{x}\right) < \frac{1}{x}\ .$$

Solution. *We first show that:*

$$\ln(1+x) < x \ .$$

Consider the function:

$$f(x) = \ln(1+x) - x \ .$$

Then, $f(0) = 0$*, and* $f'(x) = \dfrac{-x}{1+x}$*.*

Hence, $f'(x) < 0$ *for all* $x > 0$*.*

For any $x > 0$*, by the Mean Value Theorem we have:*

$$\frac{f(x) - f(0)}{x - 0} = f'(c)$$

for some $c \in (0, x)$*. Since* $c > 0$*, we have* $f'(c) < 0$*, which implies that:*

$$\frac{f(x) - f(0)}{x - 0} < 0.$$

Since $x > 0$*, we conclude that* $\ln(1+x) - x = f(x) = f(x) - f(0) < 0$*. We conclude that:*

$$\ln(1+x) < x,$$

for all $x > 0$*. To show that* $\frac{x}{1+x} < \ln(1+x)$*, we proceed similarly.*

Consider:

$$g(x) = \ln(1+x) - \frac{x}{1+x}.$$

Then, $g(0) = 0$*, and:*

$$\begin{aligned} g'(x) &= \frac{1}{1+x} - \frac{(1+x)1 - x(1)}{(1+x)^2} \\ &= \frac{x}{(1+x)^2} \\ &> 0 \end{aligned}$$

for all $x > 0$*.*

Hence, for all $x > 0$*, by the Mean Value Theorem we have:*

$$\frac{g(x) - g(0)}{x - 0} = g'(c) > 0,$$

where c *is some element which lies in* $(0, x)$*.*

This shows that $\ln(1+x) - \frac{x}{1+x} = g(x) > 0$*. Hence,*

$$\ln(1+x) > \frac{x}{1+x}$$

for $x > 0$.

Finally, for all $t > 0$*, we have* $\frac{1}{t} > 0$*. Applying the inequality:*

$$\frac{x}{1+x} < \ln(1+x) < x$$

to $x = \frac{1}{t}$*, we have:*

$$\frac{1/t}{1+1/t} < \ln\left(1+\frac{1}{t}\right) < \frac{1}{t},$$

which is equivalent to:

$$\frac{1}{t+1} < \ln\left(1+\frac{1}{t}\right) < \frac{1}{t}.$$

MATH 1010DE Week 8

Curve Sketching

8.1 Absolute/Relative (Global/Local) Extrema

Consider a function $f : A \longrightarrow \mathbb{R}$.

Definition 8.1.
- *If there is an element $c \in A$ such that: $f(c) \leq f(x)$ for all $x \in A$, we say that $f(c)$ is the (global/absolute)* **minimum** *of f.*
- *If there is an element $d \in A$ such that: $f(d) \geq f(x)$ for all $x \in A$, we say that $f(d)$ is the (global/absolute)* **maximum** *of f.*

Definition 8.2.
- *If $f(c) \leq f(x)$ for all x in an open interval containing c, we say that f has a* **local/relative minimum** *at c.*
- *If $f(c) \geq f(x)$ for all x in an open interval containing c, we say that f has a* **local/relative maximum** *at c.*

IMAGE
By KSmrq - http://commons.wikimedia.org/wiki/File:Extrema_example.svg ,
GFDL 1.2 , Link

Theorem 8.3 (First Derivative Test)**.** *Let $f : A \longrightarrow \mathbb{R}$ be a continuous function. For $c \in A$, if there exists an open interval (a, b) containing c such that $f'(x) < 0$ (in particular it exists) for all $x \in (a, c)$, and $f'(x) > 0$ for all $x \in (c, b)$, then f has a local minimum at c.*

Similarly, if $f'(x) > 0$ for all $x \in (a, c)$ and $f'(x) < 0$ for all $x \in (c, b)$, then f has a local maximum at c.

Note: In the special case that the domain of f is an open interval (a, b), if $f'(x) > 0$ for <u>all</u> $x \in (a, c)$, and $f'(x) < 0$ for <u>all</u> $x \in (c, b)$, then f has an absolute maximum at c.

Similarly f has an absolute minimum at c if each of the above inequalities is reversed.

Example 8.4.

- *In Example 7.6, the function has a local maximum at* $x = -5$*, and a local minimum at* $x = 1$.
- *In Example 7.7, the function has only one local extremum, namely a local minimum at* $x = -1$*. In fact,* $f(-1) = 0$ *is the absolute minimum of* f.

Exercise 8.5. $f(x) = x^{\frac{1}{3}} - \frac{1}{3}x - \frac{2}{3}$ *for* $x > 0$*. Show that* $f(x) \leq 0$ *for all* $x > 0$*. Then, deduce that:*

$$u^{\frac{1}{3}}v^{\frac{2}{3}} \leq \frac{1}{3}u + \frac{2}{3}v$$

for u*,* $v > 0$.

8.2 WeBWorK

1. **WeBWorK**
2. **WeBWorK**
3. **WeBWorK**
4. **WeBWorK**
5. **WeBWorK**
6. **WeBWorK**

Theorem 8.6 (Second Derivative Test)**.** *Let* f *be a function twice differentiable at* $c \in \mathbb{R}$*, such that* $f'(c) = 0$*. If:*

- $f''(c) > 0$*, then* f *has a local minimum at* c.
- $f''(c) < 0$*, then* f *has a local maximum at* c.

Proof. **Sketch of Proof.** Suppose $f''(c) > 0$, by the definition of $f''(c)$ as the derivative of f' at c, we have:

$$0 < f''(c) = \lim_{h \to 0} \frac{f'(c+h) - f'(c)}{h} = \lim_{h \to 0} \frac{f'(c+h)}{h}\ .$$

It follows from the above identity that $f'(c+h)$ is > 0 for sufficiently small positive h, and < 0 for sufficiently small negative h.

Hence there is an open interval (a, b) containing c such that f' is negative on (a, c) and positive on (c, b). So, f has a local minimum at c by the First Derivative Test.

The case $f''(c) < 0$ may be proved similarly. □

Example 8.7. *Consider the function $f(x) = x^3 + 6x^2 - 15x + 7$ in Example 7.6, we have:*

$$f''(x) = 6x + 12$$

The function f has a two stationary points $c = -5, 1$ where $f'(c) = 0$.

Since:

$$f''(-5) = -18, \quad f''(1) = 18,$$

by the Second Derivative Test $f(-5)$ is a local maximum, and $f(1)$ is a local minimum. (This corroborates the conclusions of the First Derivative Test applied to the same function, see Example 8.4.)

Example 8.8. *Consider $g(x) = x^4$. Then, $g'(x) = 4x^3$, which implies that $c = 0$ is the only point where $g'(c) = 0$.*

The second derivative of g is $g''(x) = 12x^2$. Hence, $g''(c) = g''(0) = 0$.

In this case, no conclusion can be drawn from the Second Derivative Test, regarding whether $g(0)$ is a local minimum, maximum, or neither.

However, one can still apply the First Derivative Test to conclude that $f(0) = 0$ is a local minimum.

8.3 Concavity

Let f be a twice differentiable function. If f'' is positive (resp. negative) on an open interval (a, b), then the graph of f over (a, b) is **concave up** (resp. **down**). This is due to the fact that f'' being positive (resp. negative) corresponds to f' being increasing (resp. decreasing).

IMAGE
By dino -
http://en.wikipedia.org/wiki/File:Animated_illustration_of_inflection_point.gif
Public Domain, Link

A point on the graph of f where the concavity changes is called an **inflection point**. It corresponds to a point in the domain of f where f'' changes sign.

Example 8.9. *Sketch the graph of:*

$$f(x) = \frac{x^2 + x - 2}{x^2}$$

by first finding the following information about f:

1. *Domain.*

$$\{x \in \mathbb{R} : x \neq 0\} = (-\infty, 0) \cup (0, \infty)$$

---

2. *x-intercepts (if sufficiently easy to find), and y-intercept. $f(x) = 0$ if and only if $x \neq 0$ and $x^2 + x - 2 = (x - 1)(x + 2) = 0$. Hence, the x-intercepts are:*

$$x = 1, -2.$$

In general, the y-intercept of the graph of a function is the value of the function at $x = 0$. In this case, 0 is not in the domain of f, hence the graph of f has no y-intercept.

---

3. *Asymptotes (Horizontal, Vertical, Oblique)*

$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 1.$$

Hence, the graph of f has one horizontal asymptote: $y = 1$. The value $f(x)$ is defined for all $x \neq 0$. Hence, f, being a rational function, is continuous at all $x \neq 0$. So, there are no vertical asymptotes at $x \neq 0$. Near $x = 0$, we have:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = -\infty.$$

Hence, the graph of f has a vertical asymptote at $x = 0$. Since $f(x)$ approaches 1 as x approaches $\pm\infty$, the graph of f has no oblique asymptote.

---

4. *Intervals where f is increasing/decreasing.*

$$f'(x) = \frac{x^2(2x+1)-(x^2+x-2)2x}{(x^2)^2}$$
$$= \frac{x^2(2x+1)-(x^2+x-2)2x}{(x^2)^2}$$
$$= \frac{1}{x^3}(4-x)$$

Hence, the points c where f' possibly changes sign are $c = 0, 4$.

$y = f(x)$:	$\searrow$	$\nearrow$	$\searrow$
$f'(x)$:	$-$	$+$	$-$
x:	$(-\infty, 0)$	$(0, 4)$	$(4, \infty)$

It follows from the sign chart that f is increasing on $(0, 4]$, and decreasing on $(-\infty, 0)$ and $[4, \infty)$.

---

5. *"Turning Points" on the graph of f (i.e. points corresponding to local extrema).*

It follows from the sign chart above that f as a local maximum at $x = 4$. The corresponding point on the graph is $(4, f(4)) = (4, 9/8)$.

---

6. *Intervals where f is concave up/down.*

The second derivative of f is:

$$f''(x) = \frac{1}{x^4}(2x-12)$$

The points where f'' possibly changes sign are points p where $f''(p) = 0$, or where $f''(p)$ is undefined. In this case, there are two such point: $p = 0, 6$.

$y=f(x)$:	$\cap$	$\cap$	$\cup$
$f''(x)$:	$-$	$-$	$+$
x:	$(-\infty,0)$	$(0,6)$	$(6,\infty)$

It follows from this sign chart that f is concave up on $(6,\infty)$, and concave down on $(-\infty,0)$ and $(0,6)$.

---

7. *Inflection points on the graph of f. It follows from the sign chart for f'' that $(6,f(6))=(6,10/9)$ is the only reflection point on the graph of f.*

Graph

$$y=\frac{x^2+x-2}{x^2}$$

Example 8.10. *Following the guidelines of the previous example, sketch the graph of:*

1. $f(x)=|x+1|\,(3-x)$
2. $f(x)=x+\frac{1}{|x|}$

Solution. 1. *Domain: $\mathbb{R}$. x-intercepts: $x=-1,3$. y-intercept: $y=f(0)=3$. Asymptotes: None.*

$$f'(x)=\begin{cases}2x-2 & x<-1;\\ \textit{undefined} & x=-1;\\ -2x+2 & x>-1.\end{cases}$$

Critical points: $c=-1,1$.

$$f''(x)=\begin{cases}2 & x<-1;\\ \textit{undefined} & x=-1;\\ -2 & x>-1.\end{cases}$$

Inflection point: $p=-1$.

$y = f(x)$:	$\cup \searrow$	$\cap \nearrow$	$\cap \searrow$
$f'(x)$:	$-$	$+$	$-$
$f''(x)$:	$+$	$-$	$-$
x:	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$

2. *In general, if one can rewrite a function f (e.g. using long division if f is a rational function) in the form:*

$$f(x) = mx + b + g(x),$$

such that $\lim_{x \to \pm\infty} g(x) = 0$, and m, b are constants, then one can readily conclude that $y = mx + b$ is an asymptote for the graph of f. If $m \neq 0$, we call $y = mx + b$ an oblique asymptote. If $m = 0$, then $y = b$ is a horizontal asymptote. In this example, since $\lim_{x \to \pm\infty} \frac{1}{|x|} = 0$, the graph of $f(x) = x + \frac{1}{|x|}$ has an oblique asymptote: $y = x$. We leave the rest of the calculations as an exercise. Hint :

$$f(x) = \begin{cases} x - \frac{1}{x} & x < 0; \\ x + \frac{1}{x} & x > 0. \end{cases}$$

The resulting graph is as follows:

Example 8.11. *Consider the function $f(x) = \frac{5x^2+2}{x+1}$. We have:*

$$f(x) = 5x - 5 + \frac{7}{x+1}$$

Since $\lim_{x \to \pm\infty} \frac{7}{x+1} = 0$, the graph of f approaches the line $y = 5x - 5$ as x approaches $\pm\infty$.

We conclude that $y = 5x - 5$ is an oblique asymptote for the graph of f.

In general, if the graph of f approaches the line correpsonding to $l(x) = mx + b$, as x tends to $\pm\infty$, we have:

$$m = \lim_{x \to \pm\infty} \frac{f(x)}{x}.$$

and

$$b = \lim_{x \to \pm\infty} (f(x) - mx).$$

Example 8.12. *Let* $g(x) = \sqrt{x^2 + 1} + 4$.

Exercise.

$$\lim_{x \to \infty} \frac{f(x)}{x} = 1.$$

This suggests that $y = f(x)$ *approaches* $y = x + b$ *as* $x \to \infty$, *where:*

Exercise.

$$b = \lim_{x \to \infty} (f(x) - 1 \cdot x) = 4.$$

Hence, $y = f(x)$ *approaches* $y = x + 4$ *as* x *tends to* ∞.

Similary, as x *tends to* $-\infty$, *we have:*

Exercise.

$$m = \lim_{x \to -\infty} \frac{f(x)}{x} = -1.$$

$$b = \lim_{x \to \pm -\infty} (f(x) - 1 \cdot x) = 4.$$

So, $y = f(x)$ *approaches* $y = -x + 4$ *as* x *tends to* $-\infty$.

We conclude that the graph of f *has two oblique asymptotes:*

$$y = x + 4,$$
$$y = -x + 4.$$

MATH 1010DE Week 9

L'Hôpital's Rule, Taylor Series

9.1 L'Hôpital's Rule

Theorem 9.1 (Cauchy's Mean Value Theorem)**.** *If $f, g : [a, b] \longrightarrow \mathbb{R}$ are functions which are continuous on $[a, b]$ and differentiable on (a, b), and $g(a) \neq g(b)$, then there exists $c \in (a, b)$ such that:*

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Proof. **Exercise.** Apply Rolle's Theorem to:

$$h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a))$$

□

IMAGE

Theorem 9.2 (L'Hôpital's Rule)**.** *Let $c \in \mathbb{R}$. Let $I = (a, b)$ be an open interval containing c. Let f, g be functions which are differentiable at every point in $(a, c) \cup (c, b)$. Suppose:*

- $\lim_{x \to c} f(x)$ *and* $\lim_{x \to c} g(x)$ *are both equal to* 0 *or both equal to* $\pm\infty$.
- $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ *exists.*

(• $g'(x) \neq 0$ for all $x \neq c$ in I.)

Then,

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)} .$$

Proof. (Sketch) We consider the special case where:

- $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$.
- f and g are continuous at $x = c$.

For such functions f, g, we have:

$$f(c) = g(c) = 0.$$

Hence:

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(c)}{g(x) - g(c)} = \frac{f'(t_x)}{g'(t_x)}$$

for some t_x between c and x by Cauchy's Mean Value Theorem.

As x approaches c, the element t_x lying between x and c must also approach c.

Hence, if the limit $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists, then intuitively it follows that:

$$\begin{aligned}\lim_{x\to c} \frac{f(x)}{g(x)} &= \lim_{t_x\to c} \frac{f'(t_x)}{g'(t_x)} \\ &= \lim_{t\to c} \frac{f'(t)}{g'(t)}.\end{aligned}$$

(**Optional Exercise** : To prove the above equality rigorously, one could, for example, apply the sequential criterion for the limit of a function .) □

9.1.1 Indeterminate Forms

- $\frac{0}{0}$

- $\frac{\infty}{\infty}$

- $0 \cdot \infty$

- $\infty - \infty$

- 0^0

- ∞^0

- 1^∞

Here, for example, 1^∞ represents the situation $\lim_{x\to c} f(x)^{g(x)}$ where $\lim_{x\to c} f(x) = 1$ and $\lim_{x\to c} g(x) = \infty$.

Hence, the following limit corresponds to the indeterminate form 1^∞:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x.$$

Example 9.3. *Use l'Hôpital's rule to evaluate the following limits:*

1. $\lim_{x\to 0}\frac{e^x - 1 - x - \frac{x^2}{2}}{x^3}$

Solution.

$$\lim_{x\to 0}\left(e^x - 1 - x - \frac{x^2}{2}\right) = 0.$$

$$\lim_{x\to 0} x^3 = 0.$$

$$\lim_{x\to 0}\frac{\left(e^x-1-x-\frac{x^2}{2}\right)'}{(x^3)'}=\lim_{x\to 0}\frac{e^x-1-x}{3x^2}\quad\left(\to\frac{0}{0}\right)$$

$$\begin{aligned}\lim_{x\to 0}\frac{(e^x-1-x)'}{(3x^2)'}&=\lim_{x\to 0}\frac{e^x-1}{6x}\\&=\frac{1}{6}\lim_{x\to 0}\frac{e^x-1}{x}\\&=\frac{1}{6}\end{aligned}$$

Hence, by l'Hôpital's rule,

$$\begin{aligned}\lim_{x\to 0}\frac{e^x-1-x-\frac{x^2}{2}}{x^3}&=\lim_{x\to 0}\frac{\left(e^x-1-x-\frac{x^2}{2}\right)'}{(x^3)'}\\&=\lim_{x\to 0}\frac{(e^x-1-x)'}{(3x^2)'}\\&=\frac{1}{6}\end{aligned}$$

---

2. $\lim_{x\to 0^+} x^{\frac{1}{1+\ln x}}$

Solution. *(This limit corresponds to the indeterminate form 0^0.)*

For $x>0$, we have $x=e^{\ln x}$. Hence,

$$\lim_{x\to 0^+} x^{\frac{1}{1+\ln x}}=\lim_{x\to 0^+} e^{\left(\frac{1}{1+\ln x}\right)\ln x}=e^{\lim_{x\to 0^+}\frac{\ln x}{1+\ln x}}$$

The last equality holds because $f(x)=e^x$ is a continuous function.

The limit $\lim_{x\to 0^+}\dfrac{\ln x}{1+\ln x}$ corresponds to the indeterminate form $\frac{-\infty}{-\infty}$. So, it is possible in this case to apply l'Hopital's rule to help find the limit.

$$\begin{aligned}\lim_{x\to 0^+}\frac{(\ln x)'}{(1+\ln x)'}&=\lim_{x\to 0^+}\frac{\frac{1}{x}}{\frac{1}{x}}\\&=\lim_{x\to 0^+}\frac{x}{x}\\&=1\end{aligned}$$

By l'Hopital's rule, it now follows that:

$$\lim_{x \to 0^+} \frac{\ln x}{1 + \ln x} = 1.$$

Hence,

$$\begin{aligned} \lim_{x \to 0^+} x^{\frac{1}{1+\ln x}} &= e^{\lim_{x \to 0^+} \frac{\ln x}{1+\ln x}} \\ &= e^1 \\ &= e. \end{aligned}$$

---

3. $\lim_{x \to +\infty} x \left(\frac{\pi}{2} - \tan^{-1} x \right)$

Solution. *(This limit corresponds to the indeterminate form* $\infty \cdot 0$*.) Rewrite the limit as follows:*

$$\lim_{x \to +\infty} x \left(\frac{\pi}{2} - \tan^{-1} x \right) = \lim_{x \to +\infty} \frac{\frac{\pi}{2} - \tan^{-1} x}{\frac{1}{x}} \quad \left(\to \frac{0}{0} \right).$$

Now, compute:

$$\begin{aligned} \lim_{x \to +\infty} \frac{\left(\frac{\pi}{2} - \tan^{-1} x \right)'}{\left(\frac{1}{x} \right)'} &= \lim_{x \to +\infty} \frac{-\frac{1}{1+x^2}}{-\frac{1}{x^2}} \\ &= \lim_{x \to +\infty} \frac{x^2}{1 + x^2} \\ &= \lim_{x \to +\infty} \frac{1}{\left(1 + \frac{1}{x^2} \right)} \\ &= 1 \end{aligned}$$

Hence, by l'Hopital's rule,

$$\lim_{x \to +\infty} x \left(\frac{\pi}{2} - \tan^{-1} x \right) = 1.$$

---

4. $\lim\limits_{x \to +\infty} (e^x + x)^{\frac{1}{x}}$

Solution. *(This limit corresponds to the indeterminate form ∞^0.) We have:*

$$\begin{aligned}\lim_{x \to +\infty} (e^x + x)^{\frac{1}{x}} &= \lim_{x \to +\infty} \left(e^{\ln(e^x+x)}\right)^{\frac{1}{x}} \\ &= \lim_{x \to +\infty} e^{\frac{\ln(e^x+x)}{x}} \\ &= e^{\lim\limits_{x \to +\infty} \frac{\ln(e^x+x)}{x}}\end{aligned}$$

The limit $\lim\limits_{x \to +\infty} \frac{\ln(e^x+x)}{x}$ corresponds to the indeterminate form $\frac{\infty}{\infty}$.

$$\begin{aligned}\lim_{x \to +\infty} \frac{(\ln(e^x+x))'}{(x)'} &= \lim_{x \to +\infty} \frac{e^x+1}{e^x+x} \\ &= \lim_{x \to +\infty} \frac{e^x\left(1+\frac{1}{e^x}\right)}{e^x\left(1+\frac{x}{e^x}\right)} \\ &= \lim_{x \to +\infty} \frac{1+\frac{1}{e^x}}{1+\frac{x}{e^x}} \\ &= 1.\end{aligned}$$

Hence, by l'Hopital's rule,

$$\begin{aligned}\lim_{x \to +\infty} \frac{\ln(e^x+x)}{x} &= \lim_{x \to +\infty} \frac{(\ln(e^x+x))'}{(x)'} \\ &= \lim_{x \to +\infty} \frac{e^x+1}{e^x+x} \\ &= 1.\end{aligned}$$

It now follows that:

$$\lim_{x \to +\infty} (e^x + x)^{\frac{1}{x}} = e^{\lim\limits_{x \to +\infty} \frac{\ln(e^x+x)}{x}} = e^1 = e.$$

---

5. $\lim_{x\to 0} \frac{1 - x\cot x}{x\sin x}$

Solution. *(This limit corresponds to the indeterminate form* $\frac{0}{0}$*.) Note that* $\cot x = \frac{\cos x}{\sin x}$*. Rewrite the limit as follows:*

$$\lim_{x\to 0} \frac{1 - x\cot x}{x\sin x} = \lim_{x\to 0} \frac{\sin x - x\cos x}{x\sin^2 x} \qquad \left(\to \frac{0}{0}\right)$$

One's first instinct might be to differentiate both numerator and denominator right away. But this seems unwise, since, looking further down the road, we would have to deal with an indeterminate form whose denominator is $(x\sin^2 x)' = 2x\sin x\cos x + \sin^2 x$*. Repeating the differentiation of the numerator and denominator would only make the expression more and more complicated.*

A cleverer way would be to rewrite the limit as follows:

$$\lim_{x\to 0} \frac{\sin x - x\cos x}{x\sin^2 x} = \lim_{x\to 0} \frac{\sin x - x\cos x}{x^3} \cdot \frac{x^2}{\sin^2 x}.$$

This is motivated by the observation that $\sin^2 x$ *is very close to* x^2 *when* x *is close to* 0*.*

First, we have:

$$\lim_{x\to 0} \frac{x^2}{\sin^2 x} = \left(\lim_{x\to 0} \frac{x}{\sin x}\right)^2 = 1.$$

The limit $\lim_{x\to 0} \frac{\sin x - x\cos x}{x^3}$ *corresponds to the indeterminate form* $\frac{0}{0}$*. Differentiating both numerator and denominator, we have:*

$$\begin{aligned}\lim_{x\to 0} \frac{(\sin x - x\cos x)'}{(x^3)'} &= \lim_{x\to 0} \frac{\cos x + x\sin x - \cos x}{3x^2} \\ &= \lim_{x\to 0} \frac{x\sin x}{3x^2} \\ &= \lim_{x\to 0} \frac{\sin x}{3x} \\ &= \frac{1}{3}.\end{aligned}$$

Hence, by l'Hopital's rule we have:

$$\lim_{x\to 0} \frac{\sin x - x\cos x}{x^3} = \lim_{x\to 0} \frac{(\sin x - x\cos x)'}{(x^3)'} = \frac{1}{3}.$$

It now follows that:

$$\begin{aligned}\lim_{x\to 0}\frac{1-x\cot x}{x\sin x} &= \lim_{x\to 0}\frac{\sin x - x\cos x}{x^3}\cdot\lim_{x\to 0}\frac{x^2}{\sin^2 x}\\ &= \frac{1}{3}\cdot 1\\ &= \frac{1}{3}.\end{aligned}$$

Exercise 9.4.

1. **WeBWorK**
2. **WeBWorK**
3. **WeBWorK**
4. **WeBWorK**
5. **WeBWorK**
6. **WeBWorK**
7. **WeBWorK**
8. **WeBWorK**
9. **WeBWorK**

Important Note. If $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ or $\pm\infty$, and $\lim_{x\to c}\frac{f'(x)}{g'(x)}$ does not exist, it **DOES NOT** follow that $\lim_{x\to c}\frac{f(x)}{g(x)}$ does not exist.

Example 9.5.

$$\lim_{x\to\infty}\frac{x+\sin x}{x}$$

9.2 Taylor Series

Definition 9.6. *Given a function f which is n times differentiable at a. The* **n-th Taylor polynomial of f (centered) at a** *is:*

$$P(x) = \sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Observe that:

$$P^{(k)}(a) = f^{(k)}(a),$$

for $k = 0, 1, 2, \ldots, n$.

Example 9.7. *The Taylor polynomials at $a = 0$ for various functions f are as follows:*

$f(x)$	$P(x)$
$\cos x$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!}$
$\sin x$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$
e^x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$
$\ln(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1} \frac{x^n}{n}$
$\frac{1}{1-x}$	$1 + x + x^2 + x^3 + \cdots + x^n$

Note, for example, that the 5-th and 6-th Taylor polynomials of $f(x) = \sin x$ at $x = 0$ both have degree 5. Hence, an n-th Taylor polynomial does not necessarily have degree n.

- Taylor polynomials of $f(x) = \sin x$ centered at $a = 0$.
- Taylor polynomials of $f(x) = \sin x$ centered at $a = \pi/2$.

Theorem 9.8 (Taylor's Formula)**.** *Let n be a positive integer, and $a \in \mathbb{R}$. Let f be a function which is $n+1$ times differentiable on an open interval I containing a. Let:*

$$\begin{aligned} P_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \\ &= f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 \\ &\quad + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

be the n-th Taylor polynomial of f at a. Then, for any $x \in I$, we have:

$$f(x) = P_n(x) + R_n(x),$$

where the **remainder term** $R_n(x)$ *is equal to:*

$$\frac{f^{(n+1)}(c)}{(n+1)!}\,(x-a)^{n+1}$$

for some c *between* a *and* x*.*

Note that the special case $n = 0$ is equivalent to (Lagrange's) Mean Value Theorem.

Proof. Recall that $P_n^{(k)}(a) = f^{(k)}(a)$ for $k = 0, 1, 2, \ldots, n$. Moreover, observe that $P_n^{(k)} = 0$ for $k > n$, since P_n is a polynomial of degree at most n.

Let:

$$F(x) = f(x) - P_n(x), \quad G(x) = (x-a)^{n+1}.$$

Then, $F(a) = G(a) = 0$, and by Cauchy's Mean Value Theorem (Cauchy's Mean Value Theorem.), we have:

$$\begin{aligned}\frac{f(x)-P_n(x)}{(x-a)^{n+1}} &= \frac{F(x)-F(a)}{G(x)-G(a)}\\ &= \frac{F'(x_1)}{G'(x_1)}\\ &= \frac{f'(x_1)-P_n'(x_1)}{(n+1)(x_1-a)^n}\end{aligned}$$

for some x_1 between a and x.

Now let:

$$\begin{aligned}F_1(x) = F'(x) = f'(x) - P_n'(x),\\ G_1(x) = G'(x) = (n+1)(x-a)^n.\end{aligned}$$

Repeating the same procedure carried out before, we have:

$$\frac{f'(x_1)-P_n'(x_1)}{(n+1)(x_1-a)^n} = \frac{F_1'(x)}{G_1'(x)} = \frac{f^{(2)}(x_2)-P_n^{(2)}(x_2)}{(n+1)n(x_2-a)^{n-1}}$$

for some x_2 between a and x_1. Repeating this process $n+1$ times, we have:

$$\begin{aligned}
\frac{f(x) - P_n(x)}{(x-a)^{n+1}} &= \frac{f'(x_1) - P_n'(x_1)}{(n+1)(x_1-a)^n} \\
&= \frac{f^{(2)}(x_2) - P_n^{(2)}(x_2)}{(n+1)n(x_2-a)^{n-1}} \\
&\vdots \\
&= \frac{f^{(n)}(x_n) - P_n^{(n)}(x_n)}{(n+1)n(n-1)\cdots 2(x_n-a)} \\
&= \frac{f^{(n+1)}(x_{n+1}) - 0}{(n+1)!}
\end{aligned}$$

for some x_{n+1} between a and x. Letting $c = x_{n+1}$, the theorem follows. □

Definition 9.9. *Given a function f which is infinitely differentiable at a (i.e. $f^{(k)}(a)$ is defined for $k = 0, 1, 2, 3, \ldots$). The* **Taylor series of f (centered) at** *a is the power series:*

$$\begin{aligned}
T(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \\
&= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots
\end{aligned}$$

In general, for any power series of the form $S(x) = \sum_{k=0}^{\infty} a_k(x-a)^k$, the value of S at any given $c \in \mathbb{R}$ is by definition the limit:

$$S(c) := \lim_{n\to\infty} \sum_{k=0}^{n} a_k(c-a)^k.$$

Note that this limit does not necessarily exist. If it does exist, we say that the power series S **converges** at $x = c$, otherwise we say that it **diverges** at $x = c$.

Example 9.10. *The Taylor series at $a = 0$ for various functions f are as follows:*

$f(x)$	$T(x)$
$\cos x$	$\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}$
$\sin x$	$\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$
e^x	$\sum_{k=0}^{\infty} \frac{x^k}{k!}$
$\ln(1+x)$	$\sum_{k=1}^{\infty}(-1)^{k+1} \frac{x^k}{k}$
$\frac{1}{1-x}$	$\sum_{k=0}^{\infty} x^k$

Theorem 9.11 (Binomial Series)**.** *For $\alpha \in \mathbb{R}$, $|x| < 1$,*

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \cdots,$$

where:

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!} \text{ if } k > 0, \quad \binom{\alpha}{0} = 1.$$

Example 9.12. *In particular, for $|x| < 1$, we have:*

$$\begin{aligned}\sqrt{1+x} &= (1+x)^{1/2} \\ &= 1 + \frac{1}{2}x + \frac{(1/2)(1/2-1)}{2!}x^2 + \frac{(1/2)(1/2-1)(1/2-2)}{3!}x^3 + \cdots\end{aligned}$$

Example 9.13. *The Taylor $T(x)$ series of $f(x) = e^x$ at $a = 0$ converges everywhere. Moreover, for each $x \in \mathbb{R}$, we do have:*

$$T(x) = \sum_{k=0}^{\infty} \frac{1}{k!} x^k = e^x.$$

Similarly, for all $x \in \mathbb{R}$, we have:

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = \sin x$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = \cos x$$

However,

The Taylor series of $f(x) = \ln(1+x)$ at $a = 0$ is:

$$T(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \, x^k,$$

which converges only for $x \in (-1, 1]$.

For such x we do have:

$$T(x) = f(x).$$

In particular, we have:

$$\ln 2 = \ln(1+1) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \, 1^k = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Remark. *There are functions whose Taylor series converge everywhere, but not to the functions themselves.*

MATH 1010DE Week 10

Taylor Series, Indefinite Integrals, Integration by Substitution, Integration by Parts

---

10.1 Shortcuts for Computing Taylor Series

Theorem 10.1. *Let* $S(x) = \sum_{k=0}^{\infty} a_k(x-a)^k$ *be a power series which converges on an open interval of the form* $(a-r, a+r)$*,* $r > 0$*, then the function* $S(x)$ *is differentiable on* $(a-r, a+r)$*, with*

$$\begin{aligned} S'(x) &= \sum_{k=1}^{\infty} k a_k (x-a)^{k-1} \\ &= a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots + k a_k (x-a)^{k-1} + \cdots \end{aligned}$$

for all $x \in (a-r, a+r)$*.*

Applying this theorem repeatedly, it may be shown that $S(x)$ is in fact infinitely differentiable on $(a-r, a+r)$, and its Taylor series at $x = a$ is itself. That is:

$$\frac{S^{(k)}(a)}{k!} = a_k, \quad k = 0, 1, 2, \ldots.$$

Put differently:

Corollary 10.2. *Let* f *be a function. If there is a sequence* $\{a_k\}_{k=0}^{\infty}$ *such that:*

$$f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$$

for all x *in some open interval centered at* a*, then* $\sum_{k=0}^{\infty} a_k(x-a)^k$ *is the Taylor series of* f *at* $x = a$*, with* $a_k = \dfrac{f^{(k)}(a)}{k!}$*.*

Corollary 10.3. *If:*

$$\sum_{k=0}^{\infty} a_k(x-a)^k = \sum_{k=0}^{\infty} b_k(x-a)^k$$

for all x in some open interval centered at a, then $a_k = b_k$ for all k.

Exercise 10.4. *Find the Taylor series of f at the given point a.*

$f(x)$	a
$\sin(5x)$	0
$x^3 \cos x$	0
$\sin(x-\pi)$	π
$\ln x$	1
$\frac{1}{2-x}$	0
$\frac{1}{1+x}$	0
$\frac{1}{1+x^2}$	0
$\frac{x+1}{x^2+x+1}$	0

$f(x)$	a	Series
$\sin(5x)$	0	$\sum_{k=0}^{\infty} \frac{(-1)^k 5^{2k+1}}{(2k+1)!} x^{2k+1}$
$x^3 \cos x$	0	$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k+3}$
$\sin(x-\pi)$	π	$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (x-\pi)^{2k+1}$
$\ln x$	1	$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (x-1)^k$
$\frac{1}{x}$	1	$\sum_{k=0}^{\infty} (-1)^k (x-1)^k$
$\frac{1}{1+x}$	0	$\sum_{k=0}^{\infty} (-1)^k x^k$
$\frac{1}{2-x} = \frac{1}{2} \cdot \frac{1}{1+\left(-\frac{x}{2}\right)}$	0	$\sum_{k=0}^{\infty} \frac{1}{2^{k+1}} x^k$
$\frac{1}{1+x^2}$	0	$\sum_{k=0}^{\infty} (-1)^k x^{2k}$
$\frac{1}{(1+x)^2} = -\frac{d}{dx}\left(\frac{1}{1+x}\right)$	0	$\sum_{k=1}^{\infty} (-1)^{k+1} k x^{k-1}$
$\frac{x+1}{x^2+x+1} = \frac{x+1}{x^2+x+1} \cdot \frac{1-x}{1-x}$	0	$\sum_{k=0}^{\infty} \left(x^{3k} - x^{3k+2}\right)$
$\frac{1}{(1+x)(2-x)} = \frac{1}{3}\left(\frac{1}{1+x} + \frac{1}{2-x}\right)$	0	$\sum_{k=0}^{\infty} \frac{1}{3}\left((-1)^k + \frac{1}{2^{k+1}}\right) x^k$
$\arctan x$	0	$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$

10.2 WeBWorK

1. **WeBWorK**
2. **WeBWorK**
3. **WeBWorK**
4. **WeBWorK**

It is sometimes useful to use Taylor series to find limits which involve indeterminate forms.

Example 10.5.

- $$\lim_{x\to 0} \frac{\sin x - x - x^3}{x^3}$$

- $$\lim_{x\to 0} \frac{\sin x - x\cos x}{x\sin^2 x}$$

10.3 WeBWorK

1. **WeBWorK**
2. **WeBWorK**
3. **WeBWorK**
4. **WeBWorK**

10.4 Indefinite Integrals

Definition 10.6. *If $F' = f$, we say that F is an* **antiderivative** *of f.*

If two functions F and G are both antiderivatives of f over (a, b), then $F' = G' = f$, hence:

$$(F - G)' = F' - G' = 0.$$

By a corollary of the mean value theorem, this implies that $F - G$ is a constant function on (a, b). That is, there exists $C \in \mathbb{R}$, such that $(F - G)(x) = C$ for all $x \in (a, b)$.

Put differently, if F is an antiderivative of f over (a, b), then any antiderivative of f over (a, b) has the form $F + C$ for some constant function C.

Definition 10.7. *The collection of all antiderivatives of a function f is called the* **indefinite integral** *of f, denoted by:*

$$\int f(x)\,dx.$$

We call $f(x)$ the **integrand** *of $\int f(x)\,dx$.*

If $F' = f$, we write:

$$\int f(x)dx = F + C,$$

where C denotes some arbitrary constant.

Example 10.8. *Since $\frac{d}{dx}x^2 = 2x$, we write:*

$$\int 2x\,dx = x^2 + C.$$

Note that $x^2 + 17$ is also an antiderivative of $2x$, hence it is equally valid to write:

$$\int 2x\,dx = x^2 + 17 + C.$$

10.5 Some Properties of Indefinite Integrals

- $\displaystyle\int 0\ dx = C$, where C is some constant.
- For $k \in \mathbb{R}$, we have $\displaystyle\int k\ dx = kx + C$. In particular,

  $$\int dx = \int 1\,dx = x + C.$$

- For $k \neq -1$, we have:

  $$\int x^k\ dx = \frac{x^{k+1}}{k+1} + C.$$

- $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C.$
  (This identity is not quite true. Will explain later.)
- $\displaystyle\int e^x\ dx = e^x + C.$
- $\displaystyle\int \cos x\ dx = \sin x + C.$
- $\displaystyle\int \sin x\ dx = -\cos x + C.$

- $\displaystyle\int \sec^2 x \, dx = \tan x + C.$
- $\displaystyle\int \sec x \tan x \, dx = \sec x + C.$
- $\displaystyle\int \frac{1}{1+x^2} \, dx = \arctan x + C.$

- For any functions f, g with antiderivatives F, G, respectively, we have:
$$\int (f(x) + g(x)) \; dx = F(x) + G(x) + C.$$
- For $k \in \mathbb{R}$, and any function f with antiderivative F, we have: $\displaystyle\int kf(x) \, dx = kF(x) + C.$

Observe that for any $a, b \in \mathbb{R}$, and differentiable function F, by the chain rule we have:
$$\frac{d}{dx} F(ax+b) = aF'(ax+b)$$
Hence, in general we have:
$$\int f(ax+b) \, dx = \frac{1}{a} F(ax+b) + C,$$
where F is an antiderivative of f, and C is some constant.

Example 10.9.
$$\int \sin(5x + \pi/4) \, dx = -\frac{1}{5} \cos(5x + \pi/4) + C.$$

Example 10.10.
$$\int \left(x^3 + \frac{4}{x^{1/3}} + (x+7)^9 + e^{2x+1} \right) dx$$
$$= \frac{1}{4} x^4 + 4 \left(\frac{3}{2} \right) x^{2/3} + \frac{1}{10} (x+7)^{10} + \frac{1}{2} e^{2x+1} + C.$$

Example 10.11.

$$\int \sin^2(x)\,dx = \int \left(\frac{1-\cos(2x)}{2}\right) dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right) dx$$
$$= \int \frac{1}{2}\,dx - \frac{1}{2}\int \cos(2x)dx$$
$$= \frac{x}{2} - \frac{1}{4}\sin(2x) + C$$

Similarly, it may be shown that:

$$\int \cos^2(x)\,dx = \frac{x}{2} + \frac{1}{4}\sin(2x) + C$$

10.6 Integration by Substitution

Theorem 10.12. *If $F' = f$, and g is a differentiable function, then:* $\int f(g(x))g'(x)\,dx = F(g(x)) + C$.

Proof. This is just the Chain Rule in reverse, since:

$$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x).$$

□

In Leibniz Notation, the theorem may be formulated as follows: Let $u = g(x)$, then $\frac{du}{dx} = g'(x)$, and:

$$\int f(g(x))g'(x)\,dx = \int f(u)\frac{du}{dx}\,dx$$
$$= \int f(u)\,du = F(u) + C = F(g(x)) + C.$$

Example 10.13. *Evaluate:*

- $\int x^2 e^{x^3+4}\,dx$
- $\int \frac{t}{\sqrt{t+2}}\,dt$
- $\int \tan x\,dx$
- $\int \frac{x^5+x^3+x}{x^2+1}\,dx$

10.7 WeBWorK

1. **WeBWorK**
2. **WeBWorK**
3. **WeBWorK**
4. **WeBWorK**
5. **WeBWorK**
6. **WeBWorK**
7. **WeBWorK**
8. **WeBWorK**

10.8 Integration by Parts

Let u, v be differentiable functions. Recall the Product Rule:

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

Taking the indefinite integral (with respect to x) of both sides of the above equation, we have:

$$\int \frac{d}{dx}(uv)dx = \int v\frac{du}{dx}\,dx + \int u\frac{dv}{dx}\,dx,$$

which implies that:

$$\int d(uv) = \int v\,du + \int u\,dv.$$

Hence, $\displaystyle\int u\,dv = (uv) - \int v\,du$

Example 10.14. *Evaluate:*

- $\displaystyle\int xe^{3x}\,dx$
- $\displaystyle\int x^2e^x\,dx$
- $\displaystyle\int x^5e^x\,dx$

- $\int x^5 \sin x \, dx$
- $\int \ln x \, dx$
- $\int e^x \sin x \, dx$

10.9 WeBWorK

1. **WeBWorK**
2. **WeBWorK**
3. **WeBWorK**
4. **WeBWorK**
5. **WeBWorK**
6. **WeBWorK**
7. **WeBWorK**
8. **WeBWorK**

MATH 1010DE Week 11

Indefinite Integrals, Integration of Trig. Functions, Trigonometric Substitution

---

11.1 Integration of Trigonometric Functions

We have seen that:

$$\int \sin^2 x\, dx = \frac{x}{2} - \frac{1}{4}\sin(2x) + C$$

$$\int \cos^2 x\, dx = \frac{x}{2} + \frac{1}{4}\sin(2x) + C$$

Example 11.1. *Using:*

$$\int \sec^2 x\, dx = \tan x + C,$$
$$\int \csc^2 x\, dx = -\cot x + C,$$

and the identity $1 + \tan^2 x = \sec^2 x$ *(which follows from the Pythagorean Theorem), we may evaluate:*

- $\int \tan^2 x\, dx$

  $$\begin{aligned}\int \tan^2 x\, dx &= \int (\sec^2 x - 1)\, dx \\ &= \tan x - x + C,\end{aligned}$$

  where C *represents an arbitrary constant.*

- $\int \cot^2 x \, dx$

$$\begin{aligned}\int \cot^2 x \, dx &= \int (\csc^2 x - 1) \, dx \\ &= -\cot x - x + C,\end{aligned}$$

where C represents an arbitrary constant.

To evaluate an integral of the form:

$$\int \sin^m x \cos^n x \, dx, \quad n, m \in \mathbb{N},$$

it is useful to make the following substitution:

$$u = \begin{cases} \cos x, & \text{if } m \text{ is odd,} \\ \sin x, & \text{if } n \text{ is odd,} \end{cases}$$

and then apply the Pythagorean Theorem $\cos^2 x + \sin^2 x = 1$ to rewrite the original integral as:

$$\int P(u) \, du,$$

where $P(u)$ is some polynomial in u.

Example 11.2. *Evaluate:*

$$\int \cos^5 x \sin^3 x \, dx$$

$$\int \cos^5 x \sin^3 x \, dx = \int \cos^5 x \sin^2 x (\sin x \, dx)$$

Let $u = \cos x$. *Then,* $du = \sin x \, dx$. *So,*

$$\begin{aligned}\int \cos^5 x \sin^3 x \, dx &= \int \cos^5 x \sin^2 x (\sin x \, dx) \\ &= \int u^5 (1 - u^2) du \\ &= \int \left(u^5 - u^7\right) du \\ &= \frac{1}{6} u^6 - \frac{1}{8} u^8 + C \\ &= \frac{1}{6} \cos^6 x - \frac{1}{8} \cos^8 x + C,\end{aligned}$$

where C represents an arbitrary constant.

Similarly, to evaluate integrals of the form:

$$\int \tan^m x \sec^n x \, dx, \quad m, n \in \mathbb{N},$$

it is useful to make the following substitution:

$$u = \begin{cases} \sec x, & \text{if } m \text{ is odd,} \\ \tan x, & \text{if } n \text{ is even,} \end{cases}$$

and then apply the identity $1 + \tan^2 x = \sec^2 x$ to rewrite the original integral as:

$$\int P(u) \, du,$$

where $P(u)$ is some polynomial in u.

Example 11.3. *Evaluate:* $\int \tan^3 x \sec x \, dx.$

$$\int \tan^3 x \sec x \, dx = \int \tan^2 x \sec x \tan x \, dx.$$

Let $u = \sec x$*. Then,* $du = \sec x \tan x \, dx$*, and:*

$$\begin{aligned} \int \tan^3 x \sec x \, dx &= \int \tan^2 x \sec x \tan x \, dx \\ &= \int (\sec^2 x - 1) \sec x \tan x \, dx \\ &= \int (u^2 - 1) \, du \\ &= \frac{1}{3} u^3 - u + C \\ &= \frac{1}{3} \sec^3 x - \sec x + C, \end{aligned}$$

where C *represents an arbitrary constant.*

Claim 11.4.

$$\int \sec x \, dx = \ln |\sec x + \tan x| + C,$$

where C *represents an arbitrary constant.*

Proof.

$$\begin{aligned}\int \sec x\, dx &= \int \frac{1}{\cos x}\, dx \\ &= \int \frac{\cos x}{\cos^2 x}\, dx \\ &= \int \frac{\cos x}{1 - \sin^2 x}\, dx\end{aligned}$$

Let $u = \sin x$. Then $du = \cos x\, dx$, and consequently:

$$\begin{aligned}\int \sec x\, dx &= \int \frac{1}{1-u^2}\, du \\ &= \int \frac{1}{(1-u)(1+u)}\, du \\ &= \frac{1}{2} \int \left(\frac{1}{1-u} + \frac{1}{1+u} \right) du \\ &= \frac{1}{2} \left(-\ln |1-u| + \ln |1+u| \right) + C \\ &= \frac{1}{2} \ln \left| \frac{1+u}{1-u} \right| + C \\ &= \frac{1}{2} \ln \left| \frac{(1+u)^2}{1-u^2} \right| + C \\ &= \ln \left| \frac{1+u}{\sqrt{1-u^2}} \right| + C \\ &= \ln \left| \frac{1+\sin x}{\cos x} \right| + C \\ &= \ln |\sec x + \tan x| + C,\end{aligned}$$

where C represents an arbitrary constant. □

Example 11.5. *Evaluate:* $\int \sec^3 x\, dx$. *(Hint: Consider using integration by parts.)*

$$\int \sec^3 x\, dx = \int \sec x \sec^2 x\, dx.$$

Let $U = \sec x$, $dV = \sec^2 x\, dx$. *Taking* $V = \tan x$, *it follows from the Integration* *by Parts formula that:*

$$\begin{aligned}\int \sec^3 x\,dx &= \int U dV \\ &= UV - \int V\,du \\ &= \sec x \tan x - \int \tan x \sec x \tan x\,dx \\ &= \sec x \tan x - \int \sec x \tan^2 x\,dx \\ &= \sec x \tan x - \int \sec x(\sec^2 x - 1)\,dx \\ &= \sec x \tan x - \int \left(\sec^3 x - \sec x\right)\,dx \\ &= \sec x \tan x + \ln|\sec x + \tan x| - \int \sec^3 x\,dx\end{aligned}$$

This implies that:

$$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x| + C$$

where C represents an arbitrary constant. Hence:

$$\int \sec^3 x\,dx = \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C.$$

The following identities follow directly from the angle sum formulas of the sine and cosine functions:

$$\begin{aligned}\cos x \cos y &= \frac{1}{2}\left(\cos(x+y) + \cos(x-y)\right) \\ \cos x \sin y &= \frac{1}{2}\left(\sin(x+y) - \sin(x-y)\right) \\ \sin x \sin y &= \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)\end{aligned}$$

They are useful for the evaluation of integrals such as:

Example 11.6.

$$\int \cos(3x)\sin(5x)\,dx$$

$$\begin{aligned}\int \cos(3x)\sin(5x)\,dx &= \int \frac{1}{2}\left(\sin(3x+5x)-\sin(3x-5x)\right)\,dx\\ &= \frac{1}{2}\int \left(\sin(8x)+\sin(2x)\right)\,dx\\ &= \frac{1}{2}\left(-\frac{1}{8}\cos(8x)-\frac{1}{2}\cos(2x)\right)+C,\end{aligned}$$

where C represents an arbitrary constant.

11.2 WeBWorK

1. **WeBWorK**
2. **WeBWorK**
3. **WeBWorK**
4. **WeBWorK**
5. **WeBWorK**
6. **WeBWorK**
7. **WeBWorK**
8. **WeBWorK**
9. **WeBWorK**
10. **WeBWorK**

11.3 Trigonometric Substitution

When an integrand involves $\sqrt{x^2 \pm a^2}$ or $\sqrt{a^2 - x^2}$. It is sometimes useful to make the following substitution:

- $\sqrt{x^2 + a^2}$: Let $x = a\tan\theta$.
- $\sqrt{x^2 - a^2}$: Let $x = a\sec\theta$.
- $\sqrt{a^2 - x^2}$: Let $x = a\sin\theta$.

Example 11.7. *Evaluate:* $\displaystyle\int \frac{x^3}{\sqrt{1-x^2}}\,dx$

First, we note that the domain of the integrand is $(-1,1)$.

Let $\theta = \arcsin x$. *Then* $x = \sin\theta$, $dx = \cos\theta\,d\theta$, *and:*

$$\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = |\cos\theta| = \cos\theta,$$

since $\theta = \arcsin x \in [-\pi/2, \pi/2]$ *for all* $x \in (-1,1)$.

So,

$$\begin{aligned}
\int \frac{x^3}{\sqrt{1-x^2}}\,dx &= \int \frac{\sin^3\theta}{\cos\theta}\cos\theta\,d\theta \\
&= \int \sin^3\theta\,d\theta \\
&= \int (1-\cos^2\theta)\sin\theta\,d\theta \\
&= -\int (1-\cos^2\theta)\,d(\cos\theta) \\
&= -\cos\theta + \frac{1}{3}\cos^3\theta + C \\
&= -\sqrt{1-x^2} + \frac{1}{3}(1-x^2)^{3/2} + C.
\end{aligned}$$

Example 11.8. *Evaluate:* $\displaystyle\int \frac{1}{(9+x^2)^2}\,dx$

Let $\theta = \arctan(x/3)$. *Then* $x = 3\tan\theta$, $dx = 3\sec^2\theta\,d\theta$, *and:*

$$9 + x^2 = 9 + 9\tan^2\theta = 9\sec^2\theta.$$

So,

$$\begin{aligned}
\int \frac{1}{(9+x^2)^2}\,dx &= \int \frac{1}{81\sec^4\theta} 3\sec^2\theta\,d\theta \\
&= \int \frac{1}{27\sec^2\theta}\,d\theta \\
&= \frac{1}{27}\int \cos^2\theta\,d\theta \\
&= \frac{1}{27}\left(\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right) + C \\
&= \frac{1}{27}\left(\frac{\theta}{2} + \frac{2\sin\theta\cos\theta}{4}\right) + C \\
&= \frac{1}{27}\left(\frac{\theta}{2} + \frac{2\tan\theta\cos^2\theta}{4}\right) + C
\end{aligned}$$

$$= \frac{\arctan(x/3)}{54} + \frac{\tan\left(\arctan(x/3)\right)\cos^2\left(\arctan(x/3)\right)}{54} + C$$

Now,

$$\begin{aligned}\cos^2\left(\arctan(x/3)\right) &= \frac{1}{\sec^2\left(\arctan(x/3)\right)}\\ &= \frac{1}{1+\tan^2\left(\arctan(x/3)\right)}\\ &= \frac{1}{1+(x/3)^2} = \frac{9}{9+x^2}\end{aligned}$$

Hence,

$$\begin{aligned}\int \frac{1}{(9+x^2)^2}\,dx &= \frac{\arctan(x/3)}{54} + \frac{9x}{162(9+x^2)} + C\\ &= \frac{\arctan(x/3)}{54} + \frac{x}{18(9+x^2)} + C\end{aligned}$$

Example 11.9. *Evaluate:* $\displaystyle\int \frac{\sqrt{x^2-25}}{x}\,dx$

Example 11.10. *Evaluate:* $\displaystyle\int \frac{x}{8-2x-x^2}\,dx.$

Example 11.11. *Evaluate:*

$$\int \frac{dx}{x\sqrt{x^2-1}}$$

First, we note that the domain of the integrand is $(-\infty,-1)\cup(1,\infty)$.

Let $\theta = \arccos(1/x)$.

Then, $x = \sec\theta$, $dx = \sec\theta\tan\theta\,d\theta$, *and:*

$$\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = \left|\tan\theta\right|.$$

Since:

$$\theta = \arccos(1/x) \in \begin{cases}[0,\pi/2) & \text{if } x>0,\\ (\pi/2,\pi] & \text{if } x<0,\end{cases}$$

we have:

$$\sqrt{x^2-1} = |\tan\theta| = \begin{cases}\tan\theta & \text{if } x>1,\\ -\tan\theta & \text{if } x<-1.\end{cases}$$

More succinctly, we have:

$$\sqrt{x^2-1} = \operatorname{sign}(x)\tan\theta.$$

Hence,

$$\begin{aligned}\int \frac{dx}{x\sqrt{x^2-1}} &= \int \operatorname{sign}(x)\frac{\sec\theta\tan\theta}{\sec\theta\tan\theta}d\theta \\ &= \int \operatorname{sign}(x)\,d\theta \\ &= \operatorname{sign}(x)\theta + C \\ &= \operatorname{sign}(x)\arccos(1/x) + C\end{aligned}$$

Example 11.12. *Evaluate:*

$$\int \frac{x^4}{\sqrt{9-x^2}}\,dx$$

11.4 WeBWorK

1. **WeBWorK**
2. **WeBWorK**
3. **WeBWorK**
4. **WeBWorK**
5. **WeBWorK**
6. **WeBWorK**
7. **WeBWorK**
8. **WeBWorK**
9. **WeBWorK**
10. **WeBWorK**

MATH 1010DE Week 12

Indefinite Integrals, Reduction Formulas, Partial Fractions

12.1 Reduction Formulas

Let $n \in \mathbb{N}$.

Example 12.1.

$$\underbrace{\int x^n e^x \, dx}_{I_n} = x^n e^x - n \underbrace{\int x^{n-1} e^x \, dx}_{I_{n-1}} .$$

Example 12.2. *For $n \geq 2$,*

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx.$$

Let $U = \cos^{n-1} x$, $dV = \cos x \, dx$. Then:

$$dU = -(n-1) \cos^{n-2} x \sin x \, dx, \quad V = \sin x.$$

It follows from Section 10.8 () that:

$$\begin{aligned}
\int U \, dV &= UV - \int V \, dU \\
&= \cos^{n-1} x \sin x + (n-1) \int \sin^2 x \cos^{n-2} x \, dx \\
&= \cos^{n-1} x \sin x + (n-1) \int (1 - \cos^2 x) \cos^{n-2} x \, dx \\
&= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx
\end{aligned}$$

Hence:

$$(1 + (n-1)) \int \cos^n x\, dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x\, dx.$$

Dividing both sides of the equation by n, we obtain:

$$\int \cos^n x\, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x\, dx.$$

Example 12.3. *For $n \geq 2$,*

$$\int \sin^n x\, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x\, dx.$$

Example 12.4. *For $n \geq 3$,*

$$\int \sec^n x\, dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x\, dx.$$

Example 12.5.

$$\int (\ln x)^n\, dx = x(\ln x)^n - n \int (\ln x)^{n-1}\, dx.$$

12.2 WeBWorK

1. **WeBWorK**
2. **WeBWorK**
3. **WeBWorK**

12.3 Partial Fractions

Definition 12.6. *A rational function $\frac{r}{s}$, where r, s are polynomials, is said to be* **proper** *if:*

$$\deg r < \deg s.$$

By performing long division of polynomials, any rational function $\frac{p}{q}$, where p, q are polynomials, may be expressed in the form:

$$\frac{p}{q} = g + \frac{r}{q},$$

where g is a polynomial, and $\frac{r}{q}$ is a proper rational function. Let $\frac{r}{s}$ be a proper rational function. Factor s as a product of powers of distinct irreducible factors:

$$s = \cdots (x-a)^m \cdots (\underbrace{x^2+bx+c}_{\text{irreducible i.e. } b^2-4c<0})^n \cdots .$$

Then:

Fact 12.7. *The proper rational function $\frac{r}{s}$ may be written as a sum of rational functions as follows:*

$$\begin{aligned}\frac{r}{s} &= \cdots \\ &+ \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_m}{(x-a)^m} + \cdots \\ &+ \frac{B_1x+C_1}{x^2+bx+c} + \frac{B_2x+C_2}{(x^2+bx+c)^2} + \cdots + \frac{B_nx+C_n}{(x^2+bx+c)^n} \\ &+ \cdots ,\end{aligned}$$

where the A_i, B_i, C_i are constants.

Example 12.8. $\displaystyle\int \frac{x^3-x-2}{x^2-2x}\,dx$

Performing long division for polynomials, we have:

$$\begin{aligned}\int \frac{(x^3-x-2)}{x^2-2x}dx &= \int (x+2)dx + \int \frac{3x-2}{x^2-2x}dx \\ &= \frac{1}{2}x^2 + 2x + \int \frac{3x-2}{x^2-2x}dx.\end{aligned}$$

To evaluate:

$$\int \frac{3x-2}{x^2-2x}dx,$$

we first observe that the integrand is a proper rational function. Moreover, the denominator factors as follows:

$$x^2 - 2x = x(x-2).$$

Hence, by Fact 12.7 , we have:

$$\frac{3x-2}{x^2-2x} = \frac{A}{x} + \frac{B}{x-2},$$

for some constants A and B. Clearing denominators, we see that the equation above holds if and only if:

$$3x - 2 = A(x-2) + Bx. \qquad (*)$$

Letting $x = 2$, we have:

$$3 \cdot 2 - 2 = B \cdot 2,$$

which implies that $B = 2$. Similarly, letting $x = 0$ in equation $(*)$ gives:

$$-2 = -2A,$$

which implies that $A = 1$. Hence:

$$\begin{aligned}\int \frac{3x-2}{x^2-2x} dx &= \int \left(\frac{1}{x} + \frac{2}{x-2}\right) dx \\ &= \ln|x| + 2\ln|x-2| + C,\end{aligned}$$

where C represents an arbitrary constant.

We conclude that:

$$\int \frac{(x^3-x-2)}{x^2-2x} dx = \frac{1}{2}x^2 + 2x + \ln|x| + 2\ln|x-2| + C.$$

Example 12.9. $\displaystyle\int \frac{x}{(x^2+4)(x-3)}\, dx$

First we note that the integrand is a proper rational function.

The quadratic factor $x^2 + 4$ has discriminant $0^2 - 4 \cdot 4 < 0$, hence it is irreducible.

By Fact 12.7 , we have:

$$\frac{x}{(x^2+4)(x-3)} = \frac{Ax+B}{x^2+4} + \frac{C}{x-3},$$

for some constants A, B and C. Clearing denominators, the equation above holds if and only if:

$$x = (Ax+B)(x-3) + C(x^2+4) \qquad (*)$$

Letting $x = 3$, we have:

$$3 = C \cdot 13,$$

which implies that $C = 3/13$.

Letting $x = 0$, we have:

$$0 = -3B + 4C,$$

which implies that $B = (4/3)C = 4/13$.

Finally, viewing each side of equation $(*)$ as polynomials and comparing the coefficients of x^2 on each side, we have:

$$0 = A + C,$$

which implies that $A = -C = -3/13$.

Hence:

$$\begin{aligned}
&\int \frac{x}{(x^2+4)(x-3)}\,dx \\
&= \frac{1}{13}\int \frac{-3x+4}{x^2+4}\,dx + \frac{3}{13}\int \frac{1}{x-3}\,dx \\
&= \frac{1}{13}\left(\frac{-3}{2}\int \frac{1}{x^2+4}\,d(x^2+4) + \int \frac{1}{(x/2)^2+1}\,dx\right. \\
&\quad \left. +3\int \frac{1}{x-3}\,dx\right) \\
&= \frac{1}{13}\left(\frac{-3}{2}\ln\left|x^2+4\right| + 2\arctan(x/2) + 3\ln|x-3|\right) + D,
\end{aligned}$$

where D represents an arbitrary constant.

Example 12.10. $\displaystyle\int \frac{x^3}{(x^2+x+1)(x-3)^2}\,dx$

First, we observe that:

$$\frac{x^3}{(x^2+x+1)(x-3)^2}$$

is a proper rational function. Moreover, since the discriminant of x^2+x+1 is $1^2 - 4 < 0$, this quadratic factor is irreducible. So, there exist constants A, B, C, D such that:

$$\frac{x^3}{(x^2+x+1)(x-3)^2} = \frac{Ax+B}{x^2+x+1} + \frac{C}{x-3} + \frac{D}{(x-3)^2}.$$

The equation above holds if and only if:

$$\begin{aligned}
x^3 &= (Ax+B)(x-3)^2 + C(x^2+x+1)(x-3) \\
&\quad + D(x^2+x+1).
\end{aligned} \qquad (*)$$

Letting $x = 3$*, we have:*

$$27 = 13D.$$

So, $D = 27/13$*.*

To find A, B *and* C*, we view each side of the equation* $(*)$ *as polynomials, then compare the coefficients of the* x^3, x^2, x *and constant terms respectively:*

$$x^3 : \quad 1 = A + C \tag{12.3}$$

$$x^2 : \quad 0 = -6A + B - 2C + 27/13 \tag{12.4}$$

$$x : \quad 0 = 9A - 6B - 2C + 27/13 \tag{12.5}$$

$$1 : \quad 0 = 9B - 3C + 27/13 \tag{12.6}$$

Subtracting equation (12.4) *from equation* (12.5)*, we have:*

$$0 = 15A - 7B,$$

which implies that $B = 15A/7$*. Combining this with equation* (12.3)*, we have:*

$$B = 15(1 - C)/7 = 15/7 - 15C/7.$$

It now follows from equation (12.6) *that:*

$$0 = 135/7 - 135C/7 - 3C + 27/13.$$

Hence:

$$C = \frac{162}{169}$$
$$B = \frac{15}{169}$$
$$A = \frac{7}{169}$$
$$D = \frac{27}{13}.$$

We have:

$$\int \frac{x^3}{(x^2 + x + 1)(x - 3)^2}\, dx = \int \left[\frac{7x + 15}{169\,(x^2 + x + 1)} + \frac{162}{169\,(x - 3)} + \frac{27}{13\,(x - 3)^2}\right] dx$$

$$= \int \frac{7x + 15}{169\,(x^2 + x + 1)} dx + \frac{162}{169} \int \frac{1}{(x - 3)} dx + \frac{27}{13} \int \frac{1}{(x - 3)^2} dx$$

To evaluate $\int \frac{7x+15}{169(x^2+x+1)}dx$*, we first rewrite the integral as follows:*

$$\int \frac{7x+15}{169\left(x^2+x+1\right)}dx = \frac{1}{169}\int \frac{7x+7/2-7/2+15}{x^2+x+1}dx$$

$$= \frac{1}{169}\left[\frac{7}{2}\underbrace{\int \frac{2x+1}{x^2+x+1}dx}_{\int \frac{1}{x^2+x+1}d(x^2+x+1)} + \frac{23}{2}\underbrace{\int \frac{1}{(x+1/2)^2+3/4}dx}_{\frac{4}{3}\int \frac{1}{\left((2x+1)/\sqrt{3}\right)^2+1}dx}\right]$$

$$= \frac{7}{338}\ln\left|x^2+x+1\right| + \frac{23\cdot 2}{169\cdot 3}\frac{\sqrt{3}}{2}\arctan\left((2x+1)/\sqrt{3}\right) + E$$

$$= \frac{7}{338}\ln\left|x^2+x+1\right| + \frac{23}{169\sqrt{3}}\arctan\left((2x+1)/\sqrt{3}\right) + E,$$

where E represents an arbitrary constant.

It now follows that:

$$\int \frac{x^3}{(x^2+x+1)(x-3)^2}\,dx$$
$$= \frac{7}{338}\ln\left|x^2+x+1\right| + \frac{23}{169\sqrt{3}}\arctan\left((2x+1)/\sqrt{3}\right)$$
$$+ \frac{162}{169}\ln|x-3| - \frac{27}{13}\frac{1}{x-3} + E.$$

Example 12.11. $\int \frac{8x^2}{x^4+4}\,dx$

12.4 WeBWorK

1. **WeBWorK**

2. **WeBWorK**

3. **WeBWorK**

4. **WeBWorK**

5. **WeBWorK**

6. **WeBWorK**

7. **WeBWorK**
8. **WeBWorK**
9. **WeBWorK**
10. **WeBWorK**
11. **WeBWorK**
12. **WeBWorK**
13. **WeBWorK**
14. **WeBWorK**
15. **WeBWorK**

12.5 How Does Partial Fractions Decomposition Work?

This section is optional. You don't have to study it for Math 1010.

Theorem 12.12 (Unique Factorization of Real Polynomials)**.** *Given any polynomial $f \in \mathbb{R}[x]$, that is:*

$$f = a_0 + a_1 x + ... + a_n x^n, \quad a_i \in \mathbb{R},$$

There are distinct irreducible polynomials, $p_1, p_2, \ldots, p_l$ in $\mathbb{R}[x]$, of degree at most 2, such that:

$$f = p_1^{n_1} p_2^{n_2} \cdots p_l^{n_l}$$

for some positive integers $n_1, n_2, \ldots, n_l$. Up to ordering of the irreducible factors, this factorization is unique.

Theorem 12.13 (Bézout's Identity)**.** *If f and g are two irreducible polynomials in $\mathbb{R}[x]$ with no common factors, then there exist $a, b \in \mathbb{R}[x]$ such that:*

$$af + bg = 1$$

Suppose we have a rational function $\frac{p}{q}$, where $p, q \in \mathbb{R}[x]$ have no common factors, and $\deg p < \deg q$.

By Unique Factorization of Real Polynomials , there are distinct irreducible polynomials $q_1, q_2, \ldots, q_l$, of degree at most 2, such that:

$$q = q_1^{n_1} q_2^{n_2} \cdots q_l^{n_l},$$

for some positive integers $n_1, n_2, \ldots, n_l$.

Since the polynomial $q_1^{n_1}$ has no common factors with $q_2^{n_2} \ldots q_l^{n_l}$, by Bézout's Identity there exist polynomials f, g such that:

$$f \cdot (q_2^{n_2} \cdots q_l^{n_l}) + g q_1^{n_1} = 1.$$

Hence,

$$\begin{aligned}
\frac{p}{q} &= \frac{p \cdot 1}{q} \\
&= \frac{p(f q_2^{n_2} \cdots q_l^{n_l} + g q_1^{n_1})}{q_1^{n_1} q_2^{n_2} \cdots q_1^{n_l}} \\
&= \frac{pf}{q_1^{n_1}} + \frac{pg}{q_2^{n_2} \cdots q_l^{n_l}}
\end{aligned}$$

Consider now the term: $\dfrac{pf}{q_1^{n_1}}$. By the Divison Algorithm for real polynomials, we have:

$$pf = aq_1 + r$$

for some real polynomials a, r such that $\deg r < \deg q_1$. Hence,

$$\frac{pf}{q_1^{n_1}} = \frac{aq_1 + r}{q_1^{n_1}} = \frac{a}{q_1^{n_1 - 1}} + \frac{r}{q_1^{n_1}}$$

By the same reasoning, we have:

$$\frac{a}{q_1^{n_1 - 1}} = \frac{b}{q_1^{n_1 - 2}} + \frac{s}{q_1^{n_1 - 1}}$$

for some polynomials b, s such that $\deg s < \deg q_1$.

Repeating this process, eventually we have:

$$\frac{pf}{q_1^{n_1}} = \frac{r_1}{q_1} + \frac{r_2}{q_1^2} + \cdots + \frac{r_{n_1}}{q_1^{n_1}} + a_1,$$

where $\deg r_i < \deg q_1$, and a_1 is some polynomial.

We now have:

$$\frac{p}{g} = \frac{r_1}{q_1} + \frac{r_2}{q_1^2} + \cdots + \frac{r_{n_1}}{q_1^{n_1}} + a_1 + \frac{pg}{q_2^{n_2} \cdots q_l^{n_l}}.$$

Repeating the process for the term: $\dfrac{pg}{q_2^{n_2} \cdots q_l^{n_l}}$, and then for all subsequent resulting terms of similar forms, we have:

$$\frac{p}{q} = \sum_{k=1}^{l} \sum_{j=1}^{n_k} \frac{r_{kj}}{q_k^j} + h, \tag{12.7}$$

where $\deg r_{kj} < \deg q_k$, and h is some polynomial in $\mathbb{R}[x]$.

We claim that $h = 0$.

Multiplying both sides of equation (12.7) by the polynomial q, we have:

$$p = \sum_{k=1}^{l} \sum_{j=1}^{n_k} r_{kj} \cdot \frac{q}{q_k^j} + hq \tag{12.8}$$

Since every q_k^j in the sum divides q, each $\frac{q}{q_k^j}$ is a polynomial. So, the equation above is an equality between polynomials.

By assumption, $\deg p < \deg q$. On the other hand, each term:

$$r_{kj} \cdot \frac{q}{q_k^j}$$

has degree strictly less than q, since $\deg r_{kj} < \deg q_k$.

So, if $h \neq 0$, then the right-hand side of equation (12.8) has degree $\deg h + \deg q \geq \deg q > \deg p$, contradicting the equality of the two sides.

Hence, $h = 0$. It follows that:

$$\frac{p}{q} = \sum_{k=1}^{l} \sum_{j=1}^{n_k} \frac{r_{kj}}{q_k^j}$$

12.6 t-Substitution

Example 12.14. *Evaluate:*

$$\int \frac{1}{1 + 2\cos x} \, dx$$

Let:

$$t = \tan \frac{x}{2}.$$

(Here, we are assuming that $x \in (-\pi, \pi)$).

Then,

$$x = 2 \arctan t,$$
$$dx = \frac{2}{1 + t^2} dt$$

Moreover,

by the double-angle formula for the sine function, we have:

$$\begin{aligned}\sin x &= 2\sin\frac{x}{2}\cos\frac{x}{2}\\ &= 2\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\cos^2\frac{x}{2}\\ &= \frac{2\tan\frac{x}{2}}{\sec^2\frac{x}{2}}\\ &= \frac{2t}{1+t^2}\end{aligned}$$

Similarly, by the double-angle formula for the cosine function, we have:

$$\begin{aligned}\cos x &= 1-2\sin^2\frac{x}{2}\\ &= 1-2\tan^2\frac{x}{2}\cos^2\frac{x}{2}\\ &= 1-\frac{2\tan^2\frac{x}{2}}{\sec^2\frac{x}{2}}\\ &= 1-\frac{2t^2}{1+t^2}\\ &= \frac{1-t^2}{1+t^2}\end{aligned}$$

We have:

$$\begin{aligned}\int\frac{1}{1+2\cos x}\,dx &= \int\frac{1}{1+2\left(\frac{1-t^2}{1+t^2}\right)}\frac{2}{1+t^2}\,dt\\ &= \int\frac{2}{3-t^2}\,dt\\ &= \frac{1}{\sqrt{3}}\int\left(\frac{1}{\sqrt{3}+t}+\frac{1}{\sqrt{3}-t}\right)dt\\ &= \frac{1}{\sqrt{3}}\left(\ln\left|\sqrt{3}+t\right|-\ln\left|\sqrt{3}-t\right|\right)+C\\ &= \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}+\tan\frac{x}{2}}{\sqrt{3}-\tan\frac{x}{2}}\right|+C,\end{aligned}$$

where C is an arbitrary constant.

Example 12.15. *Evaluate:*

$$\int\frac{1}{1+\sin x+\cos x}dx$$

Let $t = \tan\frac{x}{2}$. *Then:*

$$\begin{aligned} dx &= \frac{2}{1+t^2}dt \\ \sin x &= \frac{2t}{1+t^2} \\ \cos x &= \frac{1-t^2}{1+t^2} \end{aligned}$$

$$\begin{aligned} \int \frac{1}{1+\sin x+\cos x}dx &= \int \frac{\frac{2}{1+t^2}dt}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}} \\ &= \int \frac{2dt}{2+2t} = \int \frac{1}{1+t}dt \\ &= \ln|1+t| + C \\ &= \ln\left|1+\tan\frac{x}{2}\right| + C \\ &= \ln\left|1+\frac{\sin x}{1+\cos x}\right| + C, \end{aligned}$$

where C *is an arbitrary constant.*

MATH 1010DE Week 13

Definite Integrals

13.1 Motivation

Given a continuous function over a closed interval. We want to approximate the area of the region bounded by the graph of the function and the x-axis.

One way to do so is by viewing the region roughly as a union of sequence of rectangles, and then adding up the areas of these rectangles.

IMAGE
5 rectangles.

IMAGE
10 rectangles.

Intuitively, we see that the more (and smaller) rectangles are used, the more closely their union approximates the region in question.

IMAGE

Definition 13.1. *Let n be a positive integer.*

Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function on a closed interval.

Let:

$$\Delta x = \frac{b - a}{n}.$$

The **Left Riemann Sum** *of f over $[a, b]$ associated with n subintervals of equal lengths is:*

$$\begin{aligned} LS_n(f) = \sum_{k=0}^{n-1} f(a + k\Delta x)\Delta x \\ = \Delta x\Big[f(a) + f(a + \Delta x) + f(a + 2\Delta x) + \dots \\ \dots + f(a + (n-1)\Delta x)\Big] \end{aligned}$$

Each summand may be thought of as the area of the rectangle whose base is the subinterval $[a + k\Delta x, a + (k+1)\Delta x]$, and whose height is the value of f at the left endpoint of the subinterval.

IMAGE$y = f(x) f(x) \Delta x x$

Definition 13.2. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function on a closed interval. The* **definite integral** $\displaystyle\int_a^b f(x)\,dx$ *of f over $[a, b]$ is equal to the limit as n tends to infinity of the left Riemann sum defined previously. That is:*

$$\begin{aligned} \int_a^b f(x)\,dx = \lim_{n\to\infty} LS_n(f) \\ = \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=0}^{n-1} f\left(a + \frac{k(b-a)}{n}\right) \end{aligned}$$

It is an established theorem that the limit exists if f is continuous.

(In fact: One could define the definite integral in terms of the Right Riemann Sum or the Midpoint Riemann Sum. All these sums tend to same limit in the case where f is continuous.) Our eventual goal is to show that if F is an antiderivative of a continuous function f, then:

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b := F(b) - F(a).$$

- **Integration by Substitution**

$$\int_a^b f(u(x))u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du = F(u(b)) - F(u(a))$$

if F is an antiderivative of f.

- **Integration by Parts**

$$\int_a^b u(x)v'(x)dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)\,dx.$$

- **Integration by Trigonometric Substitution**

$$\int_{-3}^{3} \frac{dx}{\sqrt{3^2+x^2}} = \int_{-\pi/4}^{\pi/4} \cos\theta\sec^2\theta d\theta$$

- **Reduction Formulas**

$$\int_0^{\pi/2} \cos^n x\,dx = \left(\frac{1}{n}\cos^{n-1} x\sin x\right)\Big|_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2}\cos^{n-2} x\,dx.$$

Before we prove the main theorem, we first state a couple of preliminary results.

Fact 13.3. *For a continuous function f on $[a,b]$, we have:*

$$\int_a^a f(x)\,dx = 0.$$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

Fact 13.4. *Let f be a continuous function on an interval I. For all $a, b, c \in I$, we have:*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$$

If f is an odd continuous function, then:

$$\begin{aligned}\int_{-a}^{a} f(x)dx &= \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx \\ &= \int_{-a}^{0} -(f(-x))dx + \int_{0}^{a} f(x)dx \\ &= \underbrace{\int_{t=a}^{t=0} (f(t))dt}_{t=-x} + \int_{0}^{a} f(x)dx \\ &= \int_{a}^{a} f(x)dx \\ &= 0\end{aligned}$$

If f is an even continuous function, then:

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

Claim 13.5. *Let f, g be continuous functions on $[a, b]$. If $f(x) \leq g(x)$ for all $x \in [a, b]$, then:*

$$\int_{a}^{b} f(x)\,dx \leq \int_{a}^{b} g(x)\,dx.$$

Example 13.6. *Find the area of the region in the xy-plane bounded between the graph of $y = x^2 - 2x - 3$ and the x-axis over the interval $[1, 5]$.*

IMAGE

The geometric area of the region described is equal to:

$$\int_{1}^{5} \left|x^2 - 2x - 3\right|\,dx$$

Consider the sign chart for the values of $f(x) = x^2 - 2x - 3 = (x + 1)(x - 3)$ over the interval $[1, 5]$:

$f(x)$:	$-$	0	$+$
x:	$[1, 3)$	3	$(3, 5]$

Hence,

$$\begin{aligned}
\int_1^5 &\left|x^2-2x-3\right|\,dx \\
&= \int_1^3 \left|x^2-2x-3\right|\,dx + \int_3^5 \left|x^2-2x-3\right|\,dx \\
&= \int_1^3 -\left(x^2-2x-3\right)\,dx + \int_3^5 \left(x^2-2x-3\right)\,dx \\
&= -\left(\frac{1}{3}x^3-x^2-3x\right)\Big|_1^3 + \left(\frac{1}{3}x^3-x^2-3x\right)\Big|_3^5 \\
&= \frac{16}{3}+\frac{32}{3} \\
&= 16
\end{aligned}$$

Theorem 13.7. *(***Mean Value Theorem for Integrals***) Let f be a continuous function on $[a,b]$. There exists $c \in [a,b]$ such that:*

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Proof. Since f is continuous on $[a,b]$, by the Extreme Value Theorem it has a maximum value M and minimum value m on $[a,b]$.

In other words,

$$m \le f(x) \le M$$

for all $x \in [a,b]$. Hence:

$$\underbrace{\int_a^b m\,dx}_{m(b-a)} \le \int_a^b f(x)\,dx \le \underbrace{\int_a^b M\,dx}_{M(b-a)}.$$

Dividing each expression by $b-a$, we have:

$$m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M.$$

Let x_1, x_2 be elements in $[a,b]$ such that $M = f(x_1)$ and $m = f(x_2)$. Since f is continuous on $[a,b]$, and $\frac{1}{b-a}\int_a^b f(x)\,dx$ is a number between $f(x_1)$ and $f(x_2)$, by the Intermediate Value Theorem there exists c between x_1 and x_2 such that:

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

This c lies in $[a,b]$, since x_1, x_2 lies in $[a,b]$. □

Theorem 13.8 (Fundamental Theorem of Calculus Part I). *Let f be a continuous function on $[a, b]$. Define a function $F : [a, b] \longrightarrow \mathbb{R}$ as follows:*

$$F(x) = \int_a^x f(t)\,dt, \quad x \in [a, b].$$

Then, F is continuous on $[a, b]$ and differentiable on (a, b), with:

$$F'(x) = f(x)$$

for all $x \in (a, b)$. Equivalently:

$$\frac{d}{dx} \int_a^x f(t)\,dt = f(x)$$

Proof. By definition:

$$\begin{aligned} F'(x) &= \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}. \\ &= \lim_{h \to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}. \\ &= \lim_{h \to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}. \end{aligned}$$

By the Mean Value Theorem for Integrals, there exists $c_h \in [x, x+h]$ such that:

$$f(c_h) = \frac{\int_x^{x+h} f(t)\,dt}{h}.$$

Hence:

$$F'(x) = \lim_{h \to 0} f(c_h) = f(x),$$

since for any h the number c_h lies between x and $x + h$, and f is continuous.

We leave the proof of the continuity of F on $[a, b]$ as an exercise. □

Corollary 13.9. *Let f be a continuous function. Let g and h be differentiable functions. Then:*

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))h'(x) - f(g(x))g'(x).$$

Example 13.10. *Evaluate:*

$$\frac{d}{dx}\int_{\sin x}^{x^3+1} e^{-t^2}\,dt$$

$$\begin{aligned}\frac{d}{dx}\int_{\sin x}^{x^3+1} e^{-t^2}\,dt &= e^{\left(-(x^3+1)^2\right)}(x^3+1)' - e^{\left(-(\sin x)^2\right)}(\sin x)' \\ &= e^{\left(-(x^3+1)^2\right)}\cdot 3x^2 - e^{\left(-(\sin x)^2\right)}\cos x\end{aligned}$$

Example 13.11. *Evaluate:*

$$\lim_{h\to 0^+}\frac{1}{\ln(1+h)}\int_2^{2+h}\sqrt{t^4+1}\,dt$$

We have:

$$\lim_{h\to 0^+}\frac{1}{\ln(1+h)}\int_2^{2+h}\sqrt{t^4+1}\,dt = \lim_{h\to 0^+}\frac{\int_2^{2+h}\sqrt{t^4+1}\,dt}{\ln(1+h)} \tag{13.9}$$

Computing the limits of the numerator and denominator separately, we have:

$$\lim_{h\to 0^+}\int_2^{2+h}\sqrt{t^4+1}\,dt = \int_2^{2}\sqrt{t^4+1}\,dt = 0$$

(because $F(h) = \int_2^{2+h}\sqrt{t^4+1}\,dt$ is a continuous function by Fundamental Theorem of Calculus Part I), and:

$$\lim_{h\to 0^+}\ln(1+h) = \ln(1+0) = 0$$

(also because $f(h) = \ln(1+h)$ is a continuous function).

Hence, the limit (13.9) *corresponds to the indeterminate form $\frac{0}{0}$.*

Taking the limit of the ratio of the derivatives of the numerator and denominator, we have:

$$\begin{aligned}\lim_{h\to 0^+}\frac{\frac{d}{dh}\int_2^{2+h}\sqrt{t^4+1}\,dt}{\frac{d}{dh}\ln(1+h)} &= \lim_{h\to 0^+}\frac{\left(\sqrt{(2+h)^4+1}\right)(2+h)'}{\frac{1}{1+h}} \\ &= \lim_{h\to 0^+}(1+h)\left(\sqrt{(2+h)^4+1}\right) \\ &= \sqrt{17}.\end{aligned}$$

It now follows from l'Hôpital's rule that:

$$\lim_{h\to 0^+} \frac{1}{\ln(1+h)} \int_2^{2+h} \sqrt{t^4+1}\,dt = \sqrt{17}.$$

There is a general formula regarding derviatives of the form:

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(x,t)\,dt,$$

the discussion of which is beyond the scope of this course. However, in certain special cases, the derivative may be found using Corollary 13.9 without much further effort:

Example 13.12. *Find:*

$$\frac{d}{dx} \int_x^{3x^2} \frac{\sin(x^2 t)}{t}\,dt, \quad x > 0. \tag{13.10}$$

Again, we first view x as a constant.
Let:

$$u = x^2 t.$$

So:

$$t = \frac{u}{x^2}, \quad dt = \frac{1}{x^2} du.$$

Under this change of variable, the integral:

$$\int_{t=x}^{t=3x^2} \frac{\sin(x^2 t)}{t}\,dt$$

is equal to:

$$\int_{u=x^3}^{u=3x^4} \frac{\sin(u)}{(u/x^2)} \frac{1}{x^2}\,du = \int_{u=x^3}^{u=3x^4} \frac{\sin(u)}{u}\,du$$

It now follows from Corollary 13.9 that:

$$\begin{aligned} \frac{d}{dx} \int_{t=x}^{t=3x^2} \frac{\sin(x^2 t)}{t}\,dt &= \frac{d}{dx} \left[\int_{u=x^3}^{u=3x^4} \frac{\sin(u)}{u}\,du \right]. \\ &= \frac{\sin(3x^4)}{3x^4} \cdot 12x^3 - \frac{\sin(x^3)}{x^3} \cdot 3x^2 \\ &= \frac{4\sin(3x^4)}{x} - \frac{3\sin(x^3)}{x}. \end{aligned}$$

Theorem 13.13 (Fundamental Theorem of Calculus Part II)**.** *Let f be a continuous function on $[a, b]$. Let F be a continuous function on $[a, b]$ which is an antiderivative of f over (a, b). Then:*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof. By the Fundamental Theorem of Calculus Part I, we know that $G(x) = \int_a^x f(t)\,dt$ is also an antiderivative of f. By Lagrange's Mean Value Theorem and the continuity of F and G on $[a, b]$, for all $x \in [a, b]$ we have:

$$G(x) = F(x) + C$$

for some constant C.

Since $G(a) = \displaystyle\int_a^a f(t)\,dt = 0$, we have $C = -F(a)$.

Hence:

$$\int_a^b f(t)\,dt = G(b) = F(b) + C = F(b) - F(a).$$

□