Math 1010 Week 5

Differentiation

5.1 Derivatives

Definition 5.1. *We say that a function f is* **differentiable** *at c if the limit:*

$$f'(c) := \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists. The limit $f'(c)$, if it exists, is called the **derivative** *of f at c.*

Interactive Example

We say that a function f is **differentiable** if it is differentiable at every point in its domain.

Exercise 5.2. *Let $f(x) = |x|$. Is f differentiable at $x = 0$? If so, find $f'(0)$.*

Theorem 5.3. *If a function f is differentiable at c, then it is also continuous at c. (The converse is false in general.)*

Example 5.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by*

$$f(x) = \begin{cases} x^3 & \text{if } \; x \leq 1; \\ ax + b & \text{if } \; x > 1. \end{cases}$$

Suppose $f(x)$ differentiable at $x = 1$, find the values of a and b.

5.2 WeBWorK

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5.3 Tangent Line

If the derivative $f'(c)$ exists, then there exists a tangent line to the graph $y = f(x)$ of f at $(c, f(c))$. Moreover, the slope of the tangent line is $f'(c)$, and the tangent line is the graph of the equation:

$$y = f'(c)(x - c) + f(c).$$

Given $f : A \longrightarrow \mathbb{R}$, the correspondence $x \mapsto f'(x)$ defines the **derivative function** $f' : A' \longrightarrow \mathbb{R}$, where A' is the set of all points $c \in A$ at which f is differentiable.

5.4 Some Common Derivative Identities

$f(x)$	$f'(x)$
constant	0
$ax + b \quad (a, b \in \mathbb{R})$	a
$x^n \quad (n \in \mathbb{Z},\ n \neq 0, 1)$	nx^{n-1}
$x^r \quad (r \in \mathbb{R},\ x > 0)$	rx^{r-1}
e^x	e^x
$a^x \quad (a > 0)$	$(\ln a)a^x$
$\ln \lvert x \rvert$	$\frac{1}{x}$
$\log_a x \quad (a \neq 1,\ a > 0)$	$\frac{1}{(\ln a)x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\csc^2 x$
$\csc x$	$-\csc x \cot x$
$\arctan x$	$\frac{1}{x^2+1}$
$\arcsin x \quad (-1 < x < 1)$	$\frac{1}{\sqrt{1-x^2}}$

5.5 Leibniz Notation

If f is defined in terms of an independent variable x, we often denote $f'(x)$ by $\frac{df}{dx}$. Under this notation, for a given $c \in \mathbb{R}$ the value $f'(c)$ is denoted by:

$$\left.\frac{df}{dx}\right|_{x=c}$$

5.6 Rules of Differentiation

Let f, g be functions differentiable at $c \in \mathbb{R}$. Then:

Sum/Difference Rule

$f \pm g$ is differentiable at c, with:

$$(f \pm g)'(c) = f'(c) \pm g'(c).$$

Proof.

$$\begin{aligned}(f+g)'(c) &= \lim_{h\to 0}\frac{(f+g)(c+h)-(f+g)(c)}{h}\\ &= \lim_{h\to 0}\frac{f(c+h)+g(c+h)-f(c)-g(c)}{h}\\ &= \lim_{h\to 0}\left[\frac{f(c+h)-f(c)}{h}+\frac{g(c+h)-g(c)}{h}\right]. \qquad (*)\end{aligned}$$

Since by assumption both $f'(c) = \lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$ and $g'(c) = \lim_{h\to 0}\frac{g(c+h)-g(c)}{h}$ exist, by the sum rule for limits the expression $(*)$ is equal to:

$$f'(c) + g'(c).$$

Exercise. Show that $(f-g)'(c) = f'(c) - g'(c)$. □

Product Rule

fg is differentiable at c, with:

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c).$$

Quotient Rule

f/g is differentiable at c provided that $g(c) \neq 0$, in which case we have:

$$\left(\frac{f}{g}\right)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{[g(c)]^2}\quad.$$

5.7 Chain Rule

Theorem 5.5. *Suppose f is differentiable at c and g is differentiable at $f(c)$, then $g \circ f$ is differentiable at c, with:*

$$(g\circ f)'(c) = g'(f(c))f'(c).$$

In the Leibniz notation, the chain rule says that if f is a differentiable function of u and u is a differentiable function of x, then:

$$\frac{df}{dx} = \frac{df}{du}\,\frac{du}{dx}\,,$$

$$\left.\frac{df}{dx}\right|_{x=c} = \left.\frac{df}{du}\right|_{u=u(c)} \left.\frac{du}{dx}\right|_{x=c}$$

Exercise 5.6. *Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by*

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x^2}\right) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

Find f'.

5.8 WeBWorK

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