Math 1010 Week 10

Taylor Series, Indefinite Integrals, Integration by Substitution, Integration by Parts

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10.1 Shortcuts for Computing Taylor Series

Theorem 10.1. *Let* $S(x) = \sum_{k=0}^{\infty} a_k(x-a)^k$ *be a power series which converges on an open interval of the form* $(a-r, a+r)$, $r > 0$, *then the function* $S(x)$ *is differentiable on* $(a-r, a+r)$, *with*

$$\begin{aligned} S'(x) &= \sum_{k=1}^{\infty} ka_k(x-a)^{k-1} \\ &= a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots + ka_k(x-a)^{k-1} + \cdots \end{aligned}$$

for all $x \in (a-r, a+r)$.

Applying this theorem repeatedly, it may be shown that $S(x)$ is in fact infinitely differentiable on $(a-r, a+r)$, and its Taylor series at $x = a$ is itself. That is:

$$\frac{S^{(k)}(a)}{k!} = a_k, \quad k = 0, 1, 2, \ldots.$$

Put differently:

Corollary 10.2. *Let* f *be a function. If there is a sequence* $\{a_k\}_{k=0}^{\infty}$ *such that:*

$$f(x) = \sum_{k=0}^{\infty} a_k(x-a)^k$$

for all x *in some open interval centered at* a, *then* $\sum_{k=0}^{\infty} a_k(x-a)^k$ *is the Taylor series of* f *at* $x = a$, *with* $a_k = \frac{f^{(k)}(a)}{k!}$.

Corollary 10.3. *If:*

$$\sum_{k=0}^{\infty} a_k(x-a)^k = \sum_{k=0}^{\infty} b_k(x-a)^k$$

for all x in some open interval centered at a, then $a_k = b_k$ for all k.

Exercise 10.4. *Find the Taylor series of f at the given point a.*

$f(x)$	a
$\sin(5x)$	0
$x^3 \cos x$	0
$\sin(x-\pi)$	π
$\ln x$	1
$\dfrac{1}{2-x}$	0
$\dfrac{1}{1+x}$	0
$\dfrac{1}{1+x^2}$	0
$\dfrac{x+1}{x^2+x+1}$	0

$f(x)$	a	Series
$\sin(5x)$	0	$\sum_{k=0}^{\infty} \frac{(-1)^k 5^{2k+1}}{(2k+1)!} x^{2k+1}$
$x^3 \cos x$	0	$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k+3}$
$\sin(x-\pi)$	π	$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (x-\pi)^{2k+1}$
$\ln x$	1	$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (x-1)^k$
$\frac{1}{x}$	1	$\sum_{k=0}^{\infty} (-1)^k (x-1)^k$
$\frac{1}{1+x}$	0	$\sum_{k=0}^{\infty} (-1)^k x^k$
$\frac{1}{2-x} = \frac{1}{2} \cdot \frac{1}{1+\left(-\frac{x}{2}\right)}$	0	$\sum_{k=0}^{\infty} \frac{1}{2^{k+1}} x^k$
$\frac{1}{1+x^2}$	0	$\sum_{k=0}^{\infty} (-1)^k x^{2k}$
$\frac{1}{(1+x)^2} = -\frac{d}{dx}\left(\frac{1}{1+x}\right)$	0	$\sum_{k=1}^{\infty} (-1)^{k+1} k x^{k-1}$
$\frac{x+1}{x^2+x+1}$	0	$\sum_{k=0}^{\infty} \left(x^{3k} - x^{3k+2}\right)$
$\frac{1}{(1+x)(2-x)} = \frac{1}{3}\left(\frac{1}{1+x} + \frac{1}{2-x}\right)$	0	$\sum_{k=0}^{\infty} \frac{1}{3}\left((-1)^k + \frac{1}{2^{k+1}}\right) x^k$
$\arctan x$	0	$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$

10.2 WeBWorK

1. **WeBWorK**
2. **WeBWorK**
3. **WeBWorK**
4. **WeBWorK**

It is sometimes useful to use Taylor series to find limits which involve indeterminate forms.

Example 10.5.

- $$\lim_{x\to 0} \frac{\sin x - x - x^3}{x^3}$$

- $$\lim_{x\to 0} \frac{\sin x - x\cos x}{x\sin^2 x}$$

10.3 WeBWorK

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10.4 Indefinite Integrals

Definition 10.6. *If $F' = f$, we say that F is an* **antiderivative** *of f.*

If two functions F and G are both antiderivatives of f over (a, b), then $F' = G' = f$, hence:

$$(F - G)' = F' - G' = 0.$$

By a corollary of the mean value theorem, this implies that $F - G$ is a constant function on (a, b). That is, there exists $C \in \mathbb{R}$, such that $(F - G)(x) = C$ for all $x \in (a, b)$.

Put differently, if F is an antiderivative of f over (a, b), then any antiderivative of f over (a, b) has the form $F + C$ for some constant function C.

Definition 10.7. *The collection of all antiderivatives of a function f is called the* **indefinite integral** *of f, denoted by:*

$$\int f(x)\,dx.$$

We call $f(x)$ the **integrand** *of $\int f(x)\,dx$.*

If $F' = f$, we write:

$$\int f(x)dx = F + C,$$

where C denotes some arbitrary constant.

Example 10.8. *Since $\frac{d}{dx}\, x^2 = 2x$, we write:*

$$\int 2x\, dx = x^2 + C.$$

Note that $x^2 + 17$ is also an antiderivative of $2x$, hence it is equally valid to write:

$$\int 2x\, dx = x^2 + 17 + C.$$

10.5 Some Properties of Indefinite Integrals

- $\displaystyle\int 0\ dx = C$, where C is some constant.

- For $k \in \mathbb{R}$, we have $\displaystyle\int k\ dx = kx + C$. In particular,

$$\int dx = \int 1\, dx = x + C.$$

- For $k \neq -1$, we have:

$$\int x^k\ dx = \frac{x^{k+1}}{k+1} + C.$$

- $\displaystyle\int \frac{1}{x}\, dx = \ln|x| + C.$

  (This identity is not quite true. Will explain later.)

- $\displaystyle\int e^x\ dx = e^x + C.$

- $\displaystyle\int \cos x\ dx = \sin x + C.$

- $\displaystyle\int \sin x\ dx = -\cos x + C.$

- $\int \sec^2 x \, dx = \tan x + C$.
- $\int \sec x \tan x \, dx = \sec x + C$.
- $\int \frac{1}{1+x^2} \, dx = \arctan x + C$.

- For any functions f, g with antiderivatives F, G, respectively, we have:

$$\int (f(x) + g(x)) \, dx = F(x) + G(x) + C.$$

- For $k \in \mathbb{R}$, and any function f with antiderivative F, we have: $\int kf(x) \, dx = kF(x) + C$.

Observe that for any $a, b \in \mathbb{R}$, and differentiable function F, by the chain rule we have:

$$\frac{d}{dx} F(ax+b) = aF'(ax+b)$$

Hence, in general we have:

$$\int f(ax+b) \, dx = \frac{1}{a} F(ax+b) + C,$$

where F is an antiderivative of f, and C is some constant.

Example 10.9.

$$\int \sin(5x + \pi/4) \, dx = -\frac{1}{5} \cos(5x + \pi/4) + C.$$

Example 10.10.

$$\int \left(x^3 + \frac{4}{x^{1/3}} + (x+7)^9 + e^{2x+1} \right) dx$$
$$= \frac{1}{4} x^4 + 4 \left(\frac{3}{2} \right) x^{2/3} + \frac{1}{10} (x+7)^{10} + \frac{1}{2} e^{2x+1} + C.$$

Example 10.11.

$$\int \sin^2(x)\,dx = \int \left(\frac{1-\cos(2x)}{2}\right) dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right) dx$$
$$= \int \frac{1}{2}\,dx - \frac{1}{2}\int \cos(2x)dx$$
$$= \frac{x}{2} - \frac{1}{4}\sin(2x) + C$$

Similarly, it may be shown that:

$$\int \cos^2(x)\,dx = \frac{x}{2} + \frac{1}{4}\sin(2x) + C$$

10.6 Integration by Substitution

Theorem 10.12. *If $F' = f$, and g is a differentiable function, then:* $\int f(g(x))g'(x)\,dx = F(g(x)) + C$.

Proof. This is just the Chain Rule in reverse, since:

$$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x).$$

□

In Leibniz Notation, the theorem may be formulated as follows: Let $u = g(x)$, then $\frac{du}{dx} = g'(x)$, and:

$$\int f(g(x))g'(x)\,dx = \int f(u)\frac{du}{dx}\,dx$$
$$= \int f(u)\,du = F(u) + C = F(g(x)) + C.$$

Example 10.13. *Evaluate:*

- $\int x^2 e^{x^3+4}\,dx$
- $\int \frac{t}{\sqrt{t+2}}\,dt$
- $\int \tan x\,dx$
- $\int \frac{x^5+x^3+x}{x^2+1}\,dx$

10.7 WeBWorK

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10.8 Integration by Parts

Let u, v be differentiable functions. Recall the Product Rule:

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

Taking the indefinite integral (with respect to x) of both sides of the above equation, we have:

$$\int \frac{d}{dx}(uv)dx = \int v\frac{du}{dx}\,dx + \int u\frac{dv}{dx}\,dx,$$

which implies that:

$$\int d(uv) = \int v\,du + \int u\,dv.$$

Hence, $\displaystyle\int u\,dv = (uv) - \int v\,du$

Example 10.14. *Evaluate:*

- $\displaystyle\int xe^{3x}\,dx$
- $\displaystyle\int x^2e^{x}\,dx$
- $\displaystyle\int x^5e^{x}\,dx$

- $\int x^5 \sin x \, dx$
- $\int \ln x \, dx$
- $\int e^x \sin x \, dx$

10.9 WeBWorK

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