MATH1050 Zermelo-Fraenkel Axioms with the Axiom of Choice

1. **Why axiomatic set theory?**

   As the pioneers in set theory discovered, while set theory would be useful in helping to pin down diverse mathematical concepts, a casual use of set theory would lead to various 'paradoxes' (such as 'Russell's Paradox' and 'Cantor's Paradox').

   One way to 'make set theory safe' is to carefully select a collection of statements as axioms which allow us to use whatever is needed in set theory but restricts ourselves from running into 'paradoxes'. The Zermelo-Fraenkel Axioms with the Axiom of Choice is one such collection.

2. **Terminologies.**

   (a) The only (mathematical) objects in the discourse are sets:

   'all things are sets'.

   (b) Undefined terms/phrases: '*sets*', '*belonging to ... (as an element)*'.
   All other terms are defined in terms of these.
   We write $x \in S$ exactly when the set x belongs to the set S; equivalently we say the set x is an element of the set S.
   We write $x \notin S$ exactly when it is not true that $x \in S$.

   (c) Common sense: *equality* '='.

   This symbol possesses these properties (as in 'usual usage'):

   i. *For any object x, the statement '$x = x$' is true.*
   ii. *For any object x, y, if $x = y$ then $y = x$.*
   iii. *For any object x, y, z if ($x = y$ and $y = z$) then $x = z$.*

3. **Axiom of Existence, Axiom of Extension and Axiom of Specification.**

   (1) **Axiom of Existence**.
   There is a set.

   (2) **Axiom of Extension**.
   Suppose A, B are sets. Then the statements ($\sharp_2$), ($\flat_2$) below are equivalent to each other:

   ($\sharp_2$) *$A = B$.*
   ($\flat_2$) *For any object x, ($x \in A$ iff $x \in B$).*

   (3) **Axiom of Specification**.
   Suppose A is a set. Suppose '$P(u)$' is a predicate with variable u. Then there exists a set B such that, for any object x, the statements ($\sharp_3$), ($\flat_3$) below are equivalent:

   ($\sharp_3$) *$x \in B$.*
   ($\flat_3$) *$x \in A$ and the statement '$P(x)$' is true.*

   Remark on notations. The set B in the context of this statement is denoted by $\{x \in A : P(x)\}$.

   Remark. In this discourse, there is only one 'method of specification'. We are allowed to construct the set $\{x \in A : P(x)\}$ out of a given set A and a given predicate $P(u)$. Given a predicate $Q(u)$, We are not allowed to immediately regard as a set the collection of all conceivable objects which upon substitution into u in the predicate $Q(u)$ result in a true statements, unless further axioms allow us to do so for that specific $Q(x)$. So every time we are tempted to write $\{x : Q(x)\}$ and claim that it is a set, we have to first think of how to justify the claim.
   The first time we will encounter this situation is when we want to talk about 'unions' for sets.

4. **Existence of the empty set and non-existence of the 'universal set'.**

   Theorem (1).

   There exists some unique set E such that the statement '$((\forall x)(x \notin E))$' is true.

   Remark. Why such a 'theorem'? We want our theory not to be void: there is some object to talk about, and one such a set is the 'empty set'. All other objects in our theory is built upon this set. There is no guarantee of the existence and the uniqueness of the empty set until proven; it is not enough to say that the empty set is a set which contains no element.

   Outline of the proof of Theorem (1).

- *Existence argument.*

  There is some set, say, A, according to the Axiom of Existence.

  Define $E = \{x \in A : x \neq x\}$, according to the Axiom of Specification.

  Now verify that the statement '$((\forall x)(x \notin E))$' is true. (Fill in the details as an exercise.)

- *Uniqueness argument.*

  Pick any set B. Suppose the statement '$(\forall x)(x \notin B)$' is true. Now apply the Axiom of Extension to verify $B = E$. (Fill in the detail as an exercise.)

Remark. The existence argument in the proof of Theorem (1) is the only occassion we need the Axiom of Existence.

Remark on terminology and notation. From now on, we will refer to the set E for which the statement '$(\forall x)(x \notin E)$' is true as the empty set. We will denote it by $\emptyset$.

Theorem (2).

Suppose A be a set. Then there exists some set B such that $B \notin A$.

Remark. It follows that there is no 'universal set' that contains every conceivable objects as its elements.

Proof of Theorem (2).

Let A be a set. Define $B = \{x \in A : x \notin x\}$, according to the Axiom of Specification.

Suppose it were true that $B \in A$.

- (Case 1.) If $B \in B$ then $B \notin B$.
- (Case 2.) If $B \notin B$ then $B \in B$.

In each case, contradiction arises.

It follows that the assumption $B \in A$ is false. Hence $B \notin A$.

Further remark. Russell's Paradox is resolved.

5. **Subsets, intersections and complements.**

   Definitions.

   (a) *Let A, B be sets. We say A is a subset of B if the statement '$(\forall x)$(if $x \in A$ then $x \in B$)' holds. We write $A \subset B$.*

   (b) *Let A, B be sets. The intersection of A and B is defined to be the set $\{x \in A : x \in B\}$. It is denoted by $A \cap B$.*

   (c) *Let A, B be sets. The complement of B and A is defined to be the set $\{x \in A : x \notin B\}$. It is denoted by $A \backslash B$.*

   Remark. According to the Axiom of Specification, the respective definitions for intersection and complement for any two given sets make sense. What makes these work is that we are seeing the intersection and the complement for the two given sets as subsets of one of the two sets.

6. **Axiom of Pairing, Axiom of Union and Axiom of Power.**

   (4) **Axiom of Pairing**.

   For any objects x, y, there exists a set A such that $x \in A$ and $y \in A$.

   (5) **Axiom of Union**.

   For any set A, there exists a set B such that, for any object x, (if ($x \in S$ for some $S \in A$) then $x \in B$).

   (6) **Axiom of Power.**

   For any set A, there exists a set B such that, for any object x, if $x \subset A$ then $x \in B$.

7. **Singletons and 'double-tons'.**

   Theorem (3).

   For any objects x, y, there exists some unique set A such that $((\forall z)((z \in A)$ iff $(z = x$ or $z = y)))$.

   Remark on notation. This set A is denoted by $\{x, y\}$. We need the Axiom of Pairing for its construction, as we cannot simply write down the chain of symbols $\{z : z = x$ or $z = y\}$ and declare it to be the set whose only elements are x and y.

   Outline of the proof of Theorem (3).

   Let x, y be objects.

- *Existence argument.*

  According to the Axiom of Pairing, there is some set, say, C, such that $x \in C$ and $y \in C$.

  Define $A = \{z \in C : z = x \text{ or } z = y\}$, according to the Axiom of Specification.

  Now by definition, for any z, (($z \in A$)' is true iff '($z = x$ or $z = y$)' is true).

- *Uniqueness argument.*

  This is an exercise in the application of the Axiom of Extension.

Theorem (4).

For any object x, there exists some unique set A such that $((\forall z)((z \in A)$ iff $(z = x)))$.

Remark on notation. This set A is denoted by $\{x\}$. Such a set is called a singleton. The proof of this result relies on the previous result.

Outline of the proof of Theorem (4). For the existence question, study the set $\{x, x\}$, whose existence is guaranteed by Theorem (3). As for the uniqueness question, apply the Axiom of Extension.

Theorem (5).

$\emptyset \neq \{\emptyset\}$.

Proof of Theorem (5). Exercise.

Remark. The argument for Theorem (5) is rooted firmly into the Axioms introduced earlier and the results proved above. It is no longer something of pure rhetoric (like '$\emptyset$ *contains no element, while* $\{\emptyset\}$ *contains one element, namely, the object* $\emptyset$').

8. **Basic set operations for any two given sets.**

Theorem (6).

For any sets A, B, there exists some unique set C such that $((\forall x)(x \in C$ iff $(x \in A$ or $x \in B)))$.

Remark on notation. The set C in this statement is called the union of A, B, and is denoted by $A \cup B$. We also abuse notation to write $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$. We need the Axiom of Union for its construction.

Outline of the proof of Theorem (6).

Pick any sets A, B.

- *Existence argument.*

  By Theorem (4), we may form the set $\{A, B\}$. Denote this set by D.

  According to the Axiom of Union, there exists a set E such that (for any object x, if $x \in S$ for some $S \in D$ then $x \in E$).

  Define $C = \{x \in E : x \in A \text{ or } x \in B\}$, according to the Axiom of Specification.

  Now by definition, for any object x, '$x \in C$' holds iff '$x \in A$ or $x \in B$' is true.

- *Uniqueness argument.*

  This is an exercise in the application of the Axiom of Extension.

Further remark. We are now ready to 'recover' all the basic results on set operations that can be formulated with 'subset relation', 'intersection (of two sets)', 'union (of two sets)' and 'complement (of one set in another)' alone. The argument for these results is just like what we have seen without knowing anything about axiom set theory.

Theorem (7).

The following statements hold:

(a) *Suppose A is a set. Then $A \subset A$.*

(b) *Let A, B be sets. Suppose $A \subset B$ and $B \subset A$. Then $A = B$.*

(c) *Let A, B, C be sets. Suppose $A \subset B$ and $B \subset C$. Then $A \subset C$.*

Theorem (8).

The following statements hold:

(a) *Suppose A is a set. Then $\emptyset \subset A$, $A \subset A$, $A \cap \emptyset = \emptyset$, $A \cap A = A$, $A \backslash \emptyset = A$, $A \backslash A = \emptyset$, $A \cup A = A$, and $A \cup \emptyset = A$.*

(b) *Suppose A, B, C are sets. Then $A \cap B = B \cap A$, and $(A \cap B) \cap C = A \cap (B \cap C)$.*

(c) *Suppose A, B, C are sets. Then $A \cup B = B \cup A$, and $(A \cup B) \cup C = A \cup (B \cup C)$.*

(d) *Suppose A, B, C are sets. Then $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$, and $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.*

(e) *Suppose A, B, C be sets. Then $(A \cap B) \backslash C = (A \backslash C) \cap (B \backslash C)$, and $(A \cup B) \backslash C = (A \backslash C) \cup (B \backslash C)$.*

(f) *Suppose A, B, C be sets. Then $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$, and $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$.*

9. **Arbitrary intersections and arbitrary unions.**

Theorem (9).

For any set H, there exists some unique set J such that $((\forall x)$ $(x \in J$ iff $(x \in S$ for some $S \in H)))$.

Remark on notation. This set J is denoted by $\bigcup_{S \in H} S$, or simply $\bigcup H$. We call this set the (generalized) union of H. We may abuse notations to write this set as $\{x \mid x \in S \text{ for some } S \in H\}$.

Theorem (10).

For any set H, if $H \neq \emptyset$ then there exists some unique set K such that $((\forall x)$ $(x \in K$ iff $(x \in S$ for any $S \in H)))$.

Remark on notation. This set K is denoted by $\bigcap_{S \in H} S$, or simply $\bigcap H$. We call this set the (generalized) intersection of H. We may abuse notations to write this set as $\{x \mid x \in S \text{ for any } S \in H\}$.

Further remark. The role of the assumption '$H \neq \emptyset$' is made apparent here; if it were true $H = \emptyset$ then '$x \in S$ for any $S \in H$' would be a true statement, giving us the 'universal set', which is forbidden from being a set, according to Theorem (2).

10. **Power sets, cartesian products, relations and functions, families.**

Theorem (11).

For any set A, there exists some unique set B such that $((\forall x)(x \in B$ iff $x \subset A))$.

Proof of Theorem (11). Exercise. (Imitate what is done for the argument of Theorem (1), Theorem (3), Theorem (6). The Axiom of Power is needed.)

Remark on notation. This set B is called the power set of A, and is denoted by $\mathfrak{P}(A)$. We abuse notations to write $\mathfrak{P}(A) = \{S \mid S \subset A\}$.

Further remark. With the help of the existence of power sets, we are now ready to 'recover' all the basic results on ordered pairs, Cartesian products, functions and relations once we have defined them formally (in the same way we did casually).

Definition.

Let x, y be objects. We define the ordered pair (x, y) to be the set $\{\{x\}, \{x, y\}\}$.

Remark. To make sense of (x, y), we need Theorem (3) and Theorem (4).

Definition.

Let x, y, z be objects. We define the ordered triple (x, y, z) to be the ordered pair $((x, y), z)$.

Theorem (12).

Let x, y, z, u, v, w be objects. The following statements hold:

(a) *$(x, y) = (u, v)$ iff $(x = u$ and $y = v)$.*

(b) *$(x, y, z) = (u, v, w)$ iff $(x = u$ and $y = v$ and $z = w)$.*

Theorem (13).

Let A, B be sets. There exists some unique set C such that $((\forall z)((z \in C)$ iff $(z = (x, y)$ for some $x \in A$, $y \in B)))$.

Remark on notation. The set C in this statement is called the Cartesian product of A and B, and is denoted by $A \times B$. We abuse notations to write $A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$.

We observe that for any $x \in A$, $y \in B$, we have $\{x\}, \{x, y\} \in \mathfrak{P}(A \cup B)$, and $(x, y) = \{\{x\}, \{x, y\}\} \in \mathfrak{P}(\mathfrak{P}(A \cup B))$. Hence $A \times B$ is defined as a subset of $\mathfrak{P}(\mathfrak{P}(A \cup B))$, through an application of the Axiom of Union, the Axiom of Power and the Axiom of Specification.

Remark. We define the notions of relations, functions, equivalence relations, and partial orderings in terms of all the above, as we did casually.

Definitions.

(a) *Let H, K, L be sets. We say (H, K, L) is a relation from H to K with graph L if $L \subset H \times K$.*

(b) *Let A, B, F be sets. Suppose (A, B, F) is a relation from A to B with graph F. We say (A, B, F) is a function from A to B with graph F if the statements $(E), (U)$ below hold:*

(E): *For any $x \in A$, there exists some $y \in B$ such that $(x, y) \in F$.*

(U): *For any $x \in A$, for any $y, z \in B$, if $(x, y) \in F$ and $(x, z) \in F$ then $y = z$.*

Remark. We recover all other basic definitions and results concerned with these mathematical objects that we have already learnt. However, at some stage, the Axiom of Choice will have to creep in for the justification of some seemingly obvious statements. To make sense of the statement of the Axiom of Choice, we introduce the notion of family and some related definitions.

Definition.

Let I, S be sets. A family in S indexed by I is a function from I to S.

Remark on notation. For such a family, say, $x : I \longrightarrow S$, we usually write $x(\beta)$ as x_β, and refer to the family as $\{x_\alpha\}_{\alpha \in I}$.

Definitions.

Let M, I be sets, and $\{S_\alpha\}_{\alpha \in I}$ be a family in $\mathfrak{P}(M)$.

(a) *Suppose $I \neq \emptyset$. Then we define the intersection of $\{S_\alpha\}_{\alpha \in I}$ to be the set $\{x \in M : x \in S_\alpha$ for any $\alpha \in I\}$.*

(b) *We define the union of $\{S_\alpha\}_{\alpha \in I}$ to be the set $\{x \in M : x \in S_\alpha$ for some $\alpha \in I\}$.*

(c) *We define the Cartesian product of $\{S_\alpha\}_{\alpha \in I}$ to be the set $\{\varphi \in \mathsf{Map}(I, M) : \varphi(\alpha) \in I\}$. ($\mathsf{Map}(I, M)$ is the set of all functions from I to M.)*

Remark. Why $\mathsf{Map}(I, M)$ is a set needs to be justified. But this can be done with the help of the Axiom of Specification, the Axiom of Union and the Axiom of Power.

11. **Successors and successor sets.**

Definition.

Let x be a set. The set $x \cup \{x\}$ is called the successor of x, and is denoted by x^+.

Remark. $\emptyset^+ = \{\emptyset\}$. Write $0 = \emptyset$, $1 = \{\emptyset\}$. $(\emptyset^+)^+ = \{0, 1\}$. Write $2 = \{0, 1\}$. $((\emptyset^+)^+)^+ = \{0, 1, 2\}$. Write $3 = \{0, 1, 2\}$. $(((\emptyset^+)^+)^+)^+ = \{0, 1, 2, 3\}$. *Et cetera.* This construction of (individual) natural numbers is due to von Neumann.

Definition.

Let A be a set. We say A is a successor set if, $\emptyset \in A$ and (for any $x \in A$, $x \cup \{x\} \in A$).

12. **Axiom of Infinity.**

(7) **Axiom of Infinity.**

There is a successor set.

13. **From the empty set to all the number systems.**

Theorem (14).

There is a unique 'smallest' successor set ω in the sense that the statements below hold:

(a) *ω is a successor set.*

(b) *For any successor set M, $\omega \subset M$.*

Remark on notation. ω is called the set of all natural numbers, and its elements are called natural numbers. By definition, $\emptyset \in \omega$. We write $0 = \emptyset$ and call it zero. We write $1 = \emptyset^+$ and call it one. *Et cetera.* The existence argument in the proof of Theorem (14) relies on the Axiom of Infinity.

Theorem (15).

The following statements hold:

(a) *There exists some unique set G such that $((\forall z)(z \in G$ iff $z = (x, x \cup \{x\})$ for some $x \in \omega$).)*

(b) *Suppose $\sigma = (\omega, \omega, G)$. Then σ is an injective function, given by $\sigma(x) = x \cup \{x\}$ for any $x \in \omega$, and $\sigma(\omega) = \omega \backslash \{0\}$.*

(c) *For any $x \in \omega$, write $x^+ = x \cup \{x\}$.*

Peano's Axioms hold for ω, in the sense that the statements (P1)-(P5) *hold:*

(P1) $0 \in \omega$.

(P2) *For any $n \in \omega$, there is some unique $m \in \omega$ such that $m = n^+$.*

(P3) *For any $m, n \in \omega$, if $m^+ = n^+$ then $m = n$.*

(P4) *For any $n \in \omega$, $0 \neq n^+$.*

(P5) *Let S be a subset of ω. Suppose $0 \in S$. Also suppose that for any $n \in \omega$, if $n \in S$ then $n^+ \in S$. Then $S = \omega$.*

Remark. The statement (P1) is an immediate consequence of ω being a successor set. The statements (P2), (P3), (P4) are encoded in the fact that σ is the unique injective function from ω to ω assigning each $x \in \omega$ to $x \cup \{x\}$. The statement (P5), which is the Principle of Mathematical Induction, can be re-stated as 'ω is a 'minimal' successor set, in the sense that any successor set which is a subset of ω is ω itself': it is a consequence of ω being the 'smallest' successor set according to Theorem (14).

Further remark. The set ω is what we are used to denote by $\mathbb{N}$ in school maths.

Theorem (16).

Let $x, y \in \mathbb{N}$. The statements below are logically equivalent:

(a) $x \in y$.

(b) $x \subset y$ and $x \neq y$.

Theorem (17).

Suppose $J = \{t \in \mathbb{N}^2$: There exist some $x, y \in \mathbb{N}$ such that $t = (x, y)$ and $x \subset y\}$. Then $(\mathbb{N}, \mathbb{N}, J)$ is a well-order relation.

Remark on notation. We agree to write $x \subset y$ exactly when $x \leq y$. In this way we recover the usual ordering for natural numbers.

Theorem (18).

Let A be a set. Suppose $b \in A$ and $f : A \longrightarrow A$ be a function. Then there exists some unique function $\varphi : \mathbb{N} \longrightarrow A$ such that $\varphi(0) = b$ and $\varphi(x^+) = f(\varphi(x))$ for any $x \in \mathbb{N}$.

Remark. Theorem (18) is known as the Recursion Theorem, and is a consequence of the Principle of Mathematical Induction.

With the help of the Principle of Mathematical Induction, we define addition in $\mathbb{N}$ 'inductively'. With addition in $\mathbb{N}$ and again with the help of the Principle of Mathematical Induction, we define multiplication in $\mathbb{N}$. The usual ordering of $\mathbb{N}$ is 'encoded' in the notion of successors. Hence we have the natural number system.

With the natural number system, we may successively construct the integer system, the rational number system, the real number system and the complex number system, with the help of functions and relations.

14. **Axiom of Substitution, Axiom of Foundation and Axiom of Choice.**

(8) **Axiom of Substitution.**
Suppose A is a set. Suppose '$S(u, v)$' is a predicate (with two variables u, v) which satisfies the condition that for each object x, there exists an object y such that the statement '$S(x, y)$' is true. Then there exists a set B such that, for any object y, the statements below are equivalent to each other:

($\sharp_8$) $y \in B$.

($\flat_8$) *The statement '$S(x, y)$' is true for some $x \in A$.*

(9) **Axiom of Foundation.**
For any x, if $x \neq \emptyset$ then there is a set $y \in x$ such that $y \cap x = \emptyset$.

(10) **Axiom of Choice.**
The cartesian product of any non-empty family of non-empty sets is non-empty.

Remark.

- The Axiom of Substitution is used for 'making sense' of the set $\{y \mid S(x, y)$ is true for some $x \in A\}$.
- The Axiom of Foundation is used for preventing 'embarrassing' situations such as '$x \in x$' from arising, and preventing 'indefinitely descending chains' $\cdots \in z \in y \in x$ from appearing.
- The Axiom of Choice is used in situations where we need to construct something but do not have any explicit 'formula' for the construction.