MATH1050 More on vector spaces and linear transformations.

0. (a) The handout is a continuation of the Handouts *Linear algebra beyond systems of linear equations and manipulation of matrices*, *Spanning sets, linearly independent sets, and bases.*

   (b) The justification for the theoretical results and the claims in the concrete examples are left as exercises in the use of sets, functions and equivalence relations in *set language*.

1. **Theorem (1).**

   Let V, W be vector spaces over a field $\mathbb{F}$, and $\varphi : V \longrightarrow W$ be a linear transformation over $\mathbb{F}$.

   The statements below hold:

   (a) *Suppose U is a subspace of V over $\mathbb{F}$. Then $\varphi(U)$ is a subspace of W over $\mathbb{F}$.*

   (b) *Let U_1, U_2 be subspaces of V over $\mathbb{F}$. Suppose U_1 is a subspace of U_2 over $\mathbb{F}$. Then $\varphi(U_1)$ is a subspace of $\varphi(U_2)$ over $\mathbb{F}$.*

   (c) *Suppose U_1, U_2 are subspaces of V over $\mathbb{F}$. Then $\varphi(U_1 + U_2) = \varphi(U_1) + \varphi(U_2)$ as vector spaces over $\mathbb{F}$.*

   (d) *Suppose U_1, U_2 are subspaces of V over $\mathbb{F}$. Then $\varphi(U_1 \cap U_2)$ is a subspace of $\varphi(U_1) \cap \varphi(U_2)$ over $\mathbb{F}$.*

2. **Theorem (2).**

   Let V, W be vector spaces over a field $\mathbb{F}$, and $\varphi : V \longrightarrow W$ be a linear transformation over $\mathbb{F}$.

   The statements below hold:

   (a) *Suppose U is a subspace of W over $\mathbb{F}$. Then $\varphi^{-1}(U)$ is a subspace of V over $\mathbb{F}$.*

   (b) *Let U_1, U_2 be subspaces of W over $\mathbb{F}$. Suppose U_1 is a subspace of U_2 over $\mathbb{F}$. Then $\varphi^{-1}(U_1)$ is a subspace of $\varphi^{-1}(U_2)$ over $\mathbb{F}$.*

   (c) *Suppose U_1, U_2 are subspaces of W over $\mathbb{F}$. Then $\varphi^{-1}(U_1+U_2)=\varphi^{-1}(U_1)+\varphi^{-1}(U_2)$ as vector spaces over $\mathbb{F}$.*

   (d) *Suppose U_1, U_2 are subspaces of W over $\mathbb{F}$. Then $\varphi^{-1}(U_1 \cap U_2) = \varphi^{-1}(U_1) \cap \varphi^{-1}(U_2)$ as vector spaces over $\mathbb{F}$.*

3. **Definition.**

   Let V, W be vector spaces over a field $\mathbb{F}$, and $\varphi : V \longrightarrow W$ be a linear transformation over $\mathbb{F}$.

   The subspace $\varphi^{-1}(\{\mathbf{0}\})$ of V is called the **kernel of the linear transformation** *φ. It is denoted by $\mathcal{N}(\varphi)$.*

   Remark on terminology. The kernel of T is also called the **null space of** φ.

4. **Examples on null spaces.**

   Refer to the Handout *Linear algebra beyond systems of linear equations and manipulation of matrices* . Given that V, W are vector spaces over a field $\mathbb{F}$, and $\varphi : V \longrightarrow W$ is a linear transformation over $\mathbb{F}$, the null space of φ is the solution set of the homogeneous linear equation

   $$\varphi(\mathbf{u}) = \mathbf{0}$$

   with unknown $\mathbf{u}$ in V.

   (a) Let $\mathbb{F}$ be a field. Suppose that A is an $(m \times n)$-matrix with entries in $\mathbb{F}$.

   Recall the null space $\mathcal{N}(A)$ of the matrix A is given by $\mathcal{N}(A) = \{\mathbf{x} \in \mathbb{F}^n : A\mathbf{x} = \mathbf{0}\}$.

   Recall that the linear transformation defined by matrix multiplication from the left by A is the linear transformation $L_A : \mathbb{F}^n \longrightarrow \mathbb{F}^m$ given by $L_A(\mathbf{x}) = A\mathbf{x}$ for any $\mathbf{x} \in \mathbb{F}^n$.

   The kernel $\mathcal{N}(L_A)$ of L_A is equal to the null space $\mathcal{N}(A)$ of the matrix A.

   (b) i. Let $c \in \mathbb{R}$. Define the function $E_c : \mathbb{R}[x] \longrightarrow \mathbb{R}$ by $E_c(f) = f(c)$ for any $f(x) \in \mathbb{R}[x]$.

   E_c is a linear transformation from $\mathbb{R}[x]$ to $\mathbb{R}$.

   The kernel of E_c is $\{f(x) \in \mathbb{R}[x] : f(c) = 0\}$. According to Factor Theorem, this is $\{f(x) \in \mathbb{R}[x] : f(x)$ is divisible by $x - c\}$.

   ii. Define the function $T : \mathbb{R}[x] \longrightarrow \mathbb{R}[x]$ by $(T(f))(x) = xf(x)$ for any $f(x) \in \mathbb{R}[x]$.

   T is a linear transformation from $\mathbb{R}[x]$ to $\mathbb{R}[x]$.

   The kernel of T is $\{0\}$. (Here 0 stands for the zero polynomial.)

iii. Define the function $S : \mathbb{R}[x] \longrightarrow \mathbb{R}[x]$ by $(S(f))(x) = f(x) - f(0)$ for any $f(x) \in \mathbb{R}[x]$.
S is a linear transformation from $\mathbb{R}[x]$ to $\mathbb{R}[x]$.
The kernel of S is $\{f(x) \in \mathbb{R}[x] : f(x) \text{ is a constant polynomial}\}$.

(c) Let J be an open interval in $\mathbb{R}$.

i. Let $c \in J$. Define the function $D_c : C^1(J) \longrightarrow \mathbb{R}$ by $D_c(\varphi) = \varphi'(c)$ for any $\varphi \in C^1(J)$.
D_c is a linear transformation from $C^1(J)$ to $\mathbb{R}$.
The kernel of D_c is the set of all real-valued functions on J which are continuously differentiable on J and whose first derivatives vanish at the point c.

ii. Define the function $D : C^1(J) \longrightarrow C(J)$ by $(D(\varphi))(x) = \varphi'(x)$ for any $\varphi \in C^1(J)$ for any $x \in J$.
D is a linear transformation from $C^1(J)$ to $C(J)$.
The kernel of D is the set of all constant real-valued functions on J. (To verify this claim, you need to apply the Mean-Value Theorem.)

(d) Let J be an interval in $\mathbb{R}$.

i. Let $c \in J$.
Define the function $I_c : C(J) \longrightarrow C^1(J)$ by $I_c(\varphi)(x) = \displaystyle\int_c^x \varphi$ for any $\varphi \in C(J)$ for any $x \in J$.
I_c is a linear transformation from $C(J)$ to $C^1(J)$.
The kernel of I_c is the singleton whose only element is the zero function on J. (To verify this claim, you need to apply the Fundamental Theorem of the Calculus.)

ii. Let $c \in J$. Let $f \in C(J)$.
Define the function $T : C(J) \longrightarrow C^1(J)$ by $T(\varphi)(x) = \displaystyle\int_c^x \varphi \cdot f$ for any $\varphi \in C(J)$ for any $x \in J$.
T is a linear transformation from $C(J)$ to $C^1(J)$.
The kernel of T is the set of all real-valued functions defined on J which are continuous on J and which vanish on the set $\{t \in J : f(t) \neq 0\}$.

5. **Theorem (3).**
Let V, W be vector spaces over a field $\mathbb{F}$, and $\varphi : V \longrightarrow W$ be a linear transformation over $\mathbb{F}$.
The statements below are logically equivalent:

(a) *φ is injective.*

(b) *For any $\mathbf{x} \in V$, if $\varphi(\mathbf{x}) = \mathbf{0}$ then $\mathbf{x} = \mathbf{0}$.*

(c) $\mathcal{N}(\varphi) = \{\mathbf{0}\}$.

Remark. Theorem (3) generalizes the result about matrices and vectors below:

Suppose A is an $(m \times n)$-matrix with entries in a field $\mathbb{F}$.
Then L_A is injective iff $\mathcal{N}(A) = \{\mathbf{0}\}$.

6. **Theorem (4).**
Let V, W be vector spaces over a field $\mathbb{F}$, and $\varphi : V \longrightarrow W$ be a linear transformation over $\mathbb{F}$.

(a) *Let $\mathbf{x}, \mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k \in V$.*
Suppose $\mathbf{x}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ over $\mathbb{F}$.
Then $\varphi(\mathbf{x})$ is a linear combination of $\varphi(\mathbf{u}_1), \varphi(\mathbf{u}_2), \cdots, \varphi(\mathbf{u}_k)$ over $\mathbb{F}$.

(b) *Suppose S is a subset of V.*
Then $\varphi(\mathsf{Span}_{\mathbb{F}}(S)) = \mathsf{Span}_{\mathbb{F}}(\varphi(S))$.

(c) *Let S be a subset of V.*
Suppose S is a spanning set for V over $\mathbb{F}$.
Then $\varphi(S)$ is a spanning set for $\varphi(V)$.

Remark. Theorem (4) generalizes the result about matrices and vectors below:

Suppose A is an $(m \times n)$-matrix with entries in a field $\mathbb{F}$.

(Recall that $L_A : \mathbb{F}^n \longrightarrow \mathbb{F}^m$ is the function defined by $L_A(\mathbf{x}) = A\mathbf{x}$ for any $\mathbf{x} \in \mathbb{F}^n$.)

The statements below hold:

(a) *Let $\mathbf{x}, \mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k \in \mathbb{F}^n$.*
Suppose $\mathbf{x}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$.
Then $A\mathbf{x}$ is a linear combination of $A\mathbf{u}_1, A\mathbf{u}_2, \cdots, A\mathbf{u}_k$.

(b) *Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k \in \mathbb{F}^n$, and $U = \left[\; \mathbf{u}_1 \; \middle| \; \mathbf{u}_2 \; \middle| \; \cdots \; \middle| \; \mathbf{u}_k \; \right]$.*
Then $L_A(\mathcal{C}(U)) = \mathcal{C}(AU)$.

(c) *Let V be a subspace of $\mathbb{F}^n$, and $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k \in \mathbb{F}^n$.*
Suppose $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k\}$ is a spanning set for V over $\mathbb{F}$.
Then $\{A\mathbf{u}_1, A\mathbf{u}_2, \cdots, A\mathbf{u}_k\}$ is a spanning set for $L_A(V)$.

(d) *$L_A(\mathbb{F}^n) = \mathcal{C}(A)$.*

7. **Theorem (5).**

Let V, W be vector spaces over a field $\mathbb{F}$, and $\varphi : V \longrightarrow W$ be a linear transformation over $\mathbb{F}$.

(a) *Let S be a subset of V. Suppose S is linearly dependent over $\mathbb{F}$.*
Then $\varphi(S)$ is linear dependent over $\mathbb{F}$.

(b) *Let T be a subset of V. Suppose T is linearly independent over $\mathbb{F}$. Further suppose φ is injective.*
Then $\varphi(T)$ is linear independent over $\mathbb{F}$.

Remark. Theorem (5) generalizes the result about matrices and vectors below:

Suppose A is an $(m \times n)$-matrix with entries in a field $\mathbb{F}$.

(a) *Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ be pairwise distinct vectors in $\mathbb{F}^n$. Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ are linearly dependent.*
Then $A\mathbf{u}_1, A\mathbf{u}_2, \cdots, A\mathbf{u}_k$ are linearly dependent vectors in $\mathbb{F}^m$.

(b) *Let $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_k$ be pairwise distinct vectors in $\mathbb{F}^n$. Suppose $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_k$ are linearly independent.*
Further suppose $\mathcal{N}(A) = \{\mathbf{0}\}$.
Then $A\mathbf{w}_1, A\mathbf{w}_2, \cdots, A\mathbf{w}_k$ are linearly independent vectors in $\mathbb{F}^m$.

8. **Theorem (6).**

Let V, W be vector spaces over a field $\mathbb{F}$, and $\varphi : V \longrightarrow W$ be a linear transformation over $\mathbb{F}$.

Let B be a base for V over $\mathbb{F}$. Further suppose φ is injective.

Then $\varphi(B)$ is a base for $\varphi(V)$ over $\mathbb{F}$.

Corollary to Theorem (6).

Let V, W be vector spaces over a field $\mathbb{F}$, and $\varphi : V \longrightarrow W$ be a linear transformation of $\mathbb{F}$. Suppose φ is injective.

Then, for any subspace U of V, for any subset C of V, C is a base for U over $\mathbb{F}$ iff $\varphi(C)$ is a base for $\varphi(U)$ over $\mathbb{F}$.

In particular, for any subset B of V, B is a base for V over $\mathbb{F}$ iff $\varphi(B)$ is a base for $\varphi(V)$ over $\mathbb{F}$.

9. **Theorem (7).**

Let V, W be vector spaces over a field $\mathbb{F}$, and $\varphi : V \longrightarrow W$ be a linear transformation over $\mathbb{F}$.

Let B be a base for $\mathcal{N}(\varphi)$ over $\mathbb{F}$, and C be a base for V over $\mathbb{F}$. Suppose $B \subset C$.

Then $\varphi(C \backslash B)$ is a base for $\varphi(V)$ over $\mathbb{F}$.

10. **Theorem (8).**

Let V, W be vector spaces over a field $\mathbb{F}$, and $\varphi : V \longrightarrow W$ be a linear transformation over $\mathbb{F}$. Suppose V is finite-dimensional over $\mathbb{F}$. The statements below hold:

(a) *$\mathcal{N}(\varphi)$ is a finite-dimensional vector space over $\mathbb{F}$, and $\dim_{\mathbb{F}}(\mathcal{N}(\varphi)) \leq \dim_{\mathbb{F}}(V)$. Equality holds iff $\varphi(\mathbf{x}) = \mathbf{0}$ for any $\mathbf{x} \in V$.*

(b) *Write $k = \dim_{\mathbb{F}}(V) - \dim_{\mathbb{F}}(\mathcal{N}(\varphi))$. Suppose B is a base for $\mathcal{N}(\varphi)$ over $\mathbb{F}$.*
Then there exist some $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k \in V \backslash \mathcal{N}(\varphi)$ such that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ are pairwise distinct, $\varphi(\mathbf{u}_1), \varphi(\mathbf{u}_2), \cdots, \varphi(\mathbf{u}_k)$ are pairwise distinct, $B \cup \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k\}$ is a base for V over $\mathbb{F}$, and $\{\varphi(\mathbf{u}_1), \varphi(\mathbf{u}_2), \cdots, \varphi(\mathbf{u}_k)\}$ is a base for $\varphi(V)$ over $\mathbb{F}$.

(c) $\dim_{\mathbb{F}}(V) = \dim_{\mathbb{F}}(\mathcal{N}(\varphi)) + \dim_{\mathbb{F}}(\varphi(V))$.

Remark on terminology. The dimension of (the finite-dimensional vector space) $\varphi(V)$ over $\mathbb{F}$ is called the **rank** of the linear transformation φ.

Remark. The equality described in Statement (c) is known as the **dimension formula** (for a linear transformation whose domain is finite-dimensional). It generalizes the result about matrices and vectors below:

Suppose A is an $(m \times n)$-matrix with entries in a field $\mathbb{F}$. Then $n = \dim_{\mathbb{F}}(\mathcal{N}(A)) + \dim_{\mathbb{F}}(\mathcal{C}(A))$.

11. **Definition.**

Let V, W be vector spaces over a field $\mathbb{F}$.

(a) *Let $\varphi : V \longrightarrow W$ be a linear transformation of $\mathbb{F}$.*
φ is called a **linear isomorphism** *if φ is bijective.*

(b) *V is said to be* **isomorphic** *to W as vector spaces over $\mathbb{F}$ if there is some linear isomorphism from V to W over $\mathbb{F}$.*

Theorem (9).

Let V, W be vector spaces over a field $\mathbb{F}$, and $\varphi : V \longrightarrow W$ be linear transformation over $\mathbb{F}$.

Suppose φ is a linear isomorphism over $\mathbb{F}$. Then the inverse function $\varphi^{-1} : W \longrightarrow V$ of the bijective function φ a linear isomorphism over $\mathbb{F}$.

12. **Theorem (10).**

Let V, W be vector spaces over a field $\mathbb{F}$, and $\varphi : V \longrightarrow W$ be linear transformation over $\mathbb{F}$.

The statements below are logically equivalent:

(a) *φ is a linear isomorphism over $\mathbb{F}$.*

(b) *For any subset B of V, if B is a base for V over $\mathbb{F}$ then $\varphi(B)$ is a base for W over $\mathbb{F}$.*

(c) *For any subset C over W, if C is a base for W over $\mathbb{F}$ then $\varphi^{-1}(C)$ is a base for V over $\mathbb{F}$.*

13. **Theorem (11).**

Let V, W be vector spaces over a field $\mathbb{F}$.

Let B be a base for V over $\mathbb{F}$.

For any function $f : B \longrightarrow W$, there exists some unique linear transformation $\varphi : V \longrightarrow W$ such that $\varphi|_B = f$ as functions.

Remark on terminology. The function φ is called the linear transformation determined by **linear extension** from f.

14. **Theorem (12).**

Let V, W be vector spaces over a field $\mathbb{F}$.

The statements below are logically equivalent:

(a) *V is isomorphic to W over $\mathbb{F}$.*

(b) *For any subset B of V, if B is a base for V over $\mathbb{F}$, then there exists some injective function $f : B \longrightarrow W$ such that $f(B)$ is a base for W over $\mathbb{F}$.*

(c) *There exist some subset C of V, some subset D of W, and some bijective function $g : C \longrightarrow D$ such that C is a base for V over $\mathbb{F}$ and D is a base for W over $\mathbb{F}$.*

Remark. We tacitly assume that every vector space over a field has a base over that field.

15. **Theorem (13).**

Let V be a vector space over a field $\mathbb{F}$. Suppose V is finite- dimensional over $\mathbb{F}$. Write $n = \dim_{\mathbb{F}}(V)$.

Then the statements below hold:

(a) *Let W be a vector space over $\mathbb{F}$. Suppose V is isomorphic to W as vector spaces over $\mathbb{F}$. Then W is finite-dimensional, and $\dim_{\mathbb{F}}(W) = n$.*

(b) *V is isomorphic to $\mathbb{F}^n$ as vector spaces over $\mathbb{F}$.*

(c) *Let W be a finite-dimensional vector space over $\mathbb{F}$. Suppose $\dim_{\mathbb{F}}(W) = n$. Then V is isomorphic to W as vector spaces over $\mathbb{F}$.*

16. **Examples on linear isomorphisms and isomorphic vector spaces.**

(a) Let $\mathbb{F}$ be a field. Suppose that A is an $(n \times n)$-square matrix with entries in $\mathbb{F}$.

L_A is a linear isomorphism from $\mathbb{F}^n$ to $\mathbb{F}_n$ iff A is non-singular.

Its inverse function ${L_A}^{-1}$ is the linear transformation $L_{A^{-1}}$.

(b) Let $\mathbb{F}$ be a field.

Recall that $\mathsf{Mat}_{m\times n}(\mathbb{F})$ is a vector space over $\mathbb{F}$, of dimension mn.

A base for $\mathsf{Mat}_{m\times n}(\mathbb{F})$ over $\mathbb{F}$ is given by $\{E_{i,j}^{m,n} \mid i \in [\![1, m]\!] \text{ and } j \in [\![1, n]\!]\}$, in which each $E_{i,j}^{m,n}$ is the $(m \times n)$-matrix with entries in $\mathbb{F}$ whose (i, j)-th entry is 1 and whose other entries are all 0.

$\mathsf{Mat}_{m\times n}(\mathbb{F})$ is isomorphic to $\mathbb{F}^{mn}$ over $\mathbb{F}$ as vector space over $\mathbb{F}$.

Recall that a base for $\mathbb{F}^{mn}$ over $\mathbb{F}$ is given by $\{\mathbf{e}_k^{(mn)} \mid k \in [\![1, mn]\!]\}$.

A bijective function f from $\{E_{i,j}^{m,n} \mid i \in [\![1, m]\!] \text{ and } j \in [\![1, n]\!]\}$ to $\{\mathbf{e}_k^{(mn)} \mid k \in [\![1, mn]\!]\}$ is given by $f(E_{i,j}^{m,n}) = \mathbf{e}_{(i-1)n+j}^{(mn)}$ for any $i \in [\![1, m]\!]$, $j \in [\![1, n]\!]$.

A linear isomorphism from $\mathsf{Mat}_{m\times n}(\mathbb{F})$ to $\mathbb{F}^{mn}$ over $\mathbb{F}$ is obtained by extending f by linearity.

When $m = 2$ and $n = 3$, the bijective function f is explicitly given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

(c) Recall that $\mathsf{Map}(\mathbb{N}, \mathbb{R})$ is the set of all infinite sequences with real entries. It is a vector space over $\mathbb{R}$.

An infinite sequence $\{a_n\}_{n=0}^{\infty}$ in the reals is said to be terminating if there exists some $N \in \mathbb{N}$ such that for any $n \in \mathbb{N}$, if $n > N$ then $a_n = 0$.

We denote by $\mathsf{Map}_{00}(\mathbb{N}, \mathbb{R})$ the set of all terminating infinite sequences in $\mathbb{R}$.

$\mathsf{Map}_{00}(\mathbb{N}, \mathbb{R})$ is a subspace of $\mathsf{Map}(\mathbb{N}, \mathbb{R})$ over $\mathbb{R}$.

A base for $\mathsf{Map}_{00}(\mathbb{N}, \mathbb{R})$ is given by $D = \{\delta_j \mid j \in \mathbb{N}\}$. (Here for each $j \in \mathbb{N}$, $\delta_j : \mathbb{N} \longrightarrow \mathbb{R}$ is given by $\delta_j(x) = \begin{cases} 1 & \text{if} \quad x = j \\ 0 & \text{if} \quad x \neq j \end{cases}$.)

Recall that $\mathbb{R}[x]$ is the vector space of all polynomials with real coefficients.

A base for $\mathbb{R}[x]$ over $\mathbb{R}$ is given by $E = \{e_j(x) \mid j \in \mathbb{N}\}$. (Here, for any $j \in \mathbb{N}$, $e_j(x)$ is the polynomial x^j.)

A bijective function $f : D \longrightarrow E$ is given by $f(\delta_j) = e_j(x)$ for any $j \in \mathbb{N}$.

$\mathsf{Map}_{00}(\mathbb{N}, \mathbb{R})$ is isomorphic to $\mathbb{R}[x]$ as vector spaces over $\mathbb{R}$.

An linear isomorphism from $\mathsf{Map}_{00}(\mathbb{N}, \mathbb{R})$ to $\mathbb{R}[x]$ over $\mathbb{R}$ is obtained by extending f by linearity.

(d) Let J be an open interval in $\mathbb{R}$.

Recall that $C(J)$ is the vector space of all real-valued functions of one real variable with domain J which are continuous on J.

Also recall that $C^1(J)$ is the vector space of all real-valued functions of one real variable with domain J which are continuously differentiable on J.

Differentiation defines the linear transformation D from $C^1(J)$ to $C(J)$, given explicitly by $(D(\varphi))(x) = \varphi'(x)$ for any $\varphi \in C^1(J)$ for any $x \in J$.

Let $a \in J$.

Recall that the function $I_a : C(J) \longrightarrow C^1(J)$ defined by $I_a(\varphi)(x) = \int_a^x \psi$ for any $\psi \in C(J)$ for any $x \in J$ is a linear transformation from $C(J)$ to $C^1(J)$.

Define $C^1(J; a) = \{\varphi \in C^1(J) : \varphi(a) = 0\}$.

$C^1(J; a)$ is a vector subspace of $C^1(J)$ over $\mathbb{R}$.

It happens that $I_a(C(J)) = C^1(J; a)$. (Why?)

The restriction of D to $C^1(J; a)$ defines a linear transformation from $C^1(J; a)$ to $C(J)$. Denote this linear transformation by Δ_a. Hence by definition, $\Delta_a(\varphi) = D(\varphi)$ for any $\varphi \in C^1(J; a)$.

Also, for any $\varphi \in C^1(J; a)$, $I_a(D(\varphi)) = \varphi$.

Moreover, for any $\psi \in C(J)$, $D(I_a(\psi)) = \psi$.

It follows that Δ_a is a linear isomorphism from $C^1(J; a)$ to $C(J)$, with its inverse function Δ_a given explicitly by ${\Delta_a}^{-1}(\psi) = I_a(\psi)$ for any $\psi \in C(J)$.

17. **Theorem (14).**

Let V be a vector space over a field $\mathbb{F}$, and U be a subspace of V over $\mathbb{F}$.

Define $E(V, U) = \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x} - \mathbf{y} \in U\}$, and $R(V, U) = (V, V, E(V, U))$.

Then $R(V, U)$ is an equivalence relation in V.

Remarks on terminologies.

(a) The equivalence relation $R(V, U)$ is called the **equivalence relation in V induced by the subspace U over $\mathbb{F}$**.

(b) The quotient in V by $R(V, U)$ is denoted by V/U, and is called the **quotient space of the vector space V by the subspace U over $\mathbb{F}$**. (We are going to define a 'natural' vector space structure on the set V/U.)

(c) For any $\mathbf{x} \in V$, the equivalence class of $\mathbf{x}$ under $R(V, U)$ is denoted by $\mathbf{x} + U$.

By definition, $\mathbf{x} + U = \{\mathbf{y} \in V : \mathbf{y} = \mathbf{x} + \mathbf{z}$ for some $\mathbf{z} \in U\}$. (This set equality needs to be verified.)

(d) The **quotient mapping from V to V/U** refers to the quotient mapping from V to V/U induced by $R(V, U)$, given by $\mathbf{x} \longmapsto \mathbf{x} + U$ for any $\mathbf{x} \in V$. It is denoted by $q_{V,U}$.

18. **Theorem (15).**

Let V be a vector space over a field $\mathbb{F}$, and U be a subspace of V over $\mathbb{F}$.

(a) *Define* $G_\Sigma = \left\{ ((J, K), L) \;\middle|\; \begin{array}{l} J, K, L \in V/U \text{ and} \\ \text{there exists some } \mathbf{x}, \mathbf{y} \in V \text{ such that} \\ J = \mathbf{x} + U,\ K = \mathbf{y} + U, \text{ and } L = (\mathbf{x} + \mathbf{y}) + U \end{array} \right\}$, *and* $\Sigma = ((V/U)^2, V/U, G_\Sigma)$.

Then Σ is a function from $(V/U)^2$ to V/U, with graph G_Σ.

Moreover, for any $\mathbf{x}, \mathbf{y} \in V$, $\Sigma(\mathbf{x} + U, \mathbf{y} + U) = (\mathbf{x} + \mathbf{y}) + U$.

(b) *Define* $G_\Pi = \left\{ ((\alpha, J), K) \;\middle|\; \begin{array}{l} \alpha \in \mathbb{F} \text{ and } J, K \in V/U \text{ and} \\ \text{there exists some } \mathbf{x} \in V \text{ such that} \\ J = \mathbf{x} + U \text{ and } K = (\alpha\mathbf{x}) + U \end{array} \right\}$, *and* $\Pi = (\mathbb{F} \times (V/U), V/U, G_\Pi)$.

Then Π is a function from $\mathbb{F} \times (V/U)$ to V/U, with graph G_Π.

Moreover, for any $\alpha \in \mathbb{F}$, for any $\mathbf{x} \in V$, $\Pi(\alpha, \mathbf{x} + U) = (\alpha\mathbf{x}) + U$.

(c) *V/U is a vector space over $\mathbb{F}$ with vector addition Σ and scalar multiplication Π.*

(d) *The quotient mapping $q_{V,U} : V \longrightarrow V/U$ is a surjective linear transformation, and $\mathcal{N}(q_{V,U}) = U$.*

Remarks on terminologies and notations.

(a) We call Σ the **(vector) addition in (the quotient space) V/U.** (Note that Σ is a closed binary operation on V/U.)

From now on we agree to write $\Sigma(J, K)$ as $J + K$ for any $J, K \in V/U$.

Hence for any $\mathbf{x}, \mathbf{y} \in V$, we have $(\mathbf{x} + U) + (\mathbf{y} + U) = (\mathbf{x} + \mathbf{y}) + U$.

(b) We call Π the **scalar multiplication in (the quotient space) V/U over (the field) $\mathbb{F}$.**

From now on we agree to write $\Pi(\alpha, J)$ as αJ for any $\alpha \in \mathbb{F}$, for any $J \in V/U$.

Hence for any $\alpha \in \mathbb{F}$, for any $\mathbf{x} \in V$, we have $\alpha(\mathbf{x} + U) = (\alpha\mathbf{x}) + U$.

19. **Theorem (16).**

Let V, W be vector spaces over a field $\mathbb{F}$, and $\varphi : V \longrightarrow W$ be a linear transformation. The statements below hold:

(a) *For any $\mathbf{x} \in V$, $\mathbf{x} + \mathcal{N}(\varphi) = \varphi^{-1}(\{\varphi(\mathbf{x})\})$.*

(b) *For any $\alpha, \beta \in \mathbb{F}$, for any $\mathbf{y}, \mathbf{z} \in \varphi(V)$, $\varphi^{-1}(\alpha\mathbf{y} + \beta\mathbf{z}) = \alpha\varphi^{-1}(\{\mathbf{y}\}) + \beta\varphi^{-1}(\{\mathbf{z}\})$.*

(c) *$V/\mathcal{N}(\varphi)$ is isomorphic to $\varphi(V)$ as vector spaces over $\mathbb{F}$ and a linear isomorphism Υ_φ from $\varphi(V)$ to $V/\mathcal{N}(\varphi)$ is given by $\Upsilon_\varphi : \mathbf{y} \longmapsto \varphi^{-1}(\{\mathbf{y}\})$ for any $\mathbf{y} \in \mathcal{N}(\varphi)$.*

The equality $q_{V,\mathcal{N}(\varphi)} = \Upsilon_\varphi \circ \varphi$ holds.

20. **Example on quotient vector spaces: Vector space of solution sets for systems of linear equations with the same coefficient matrix.**

Let $\mathbb{F}$ be a field. Suppose that A is an $(m \times n)$-matrix with entries in a field $\mathbb{F}$.

Recall that the linear transformation $L_A : \mathbb{F}^n \longrightarrow \mathbb{F}^m$ is given by $L_A(\mathbf{x}) = A\mathbf{x}$ for any $\mathbf{x} \in \mathbb{F}^n$.

Recall that the kernel of L_A is the null space $\mathcal{N}(A)$ of the matrix A, and is the solution space of the homogeneous equation $A\mathbf{u} = \mathbf{0}$ with unknown $\mathbf{u} \in \mathbb{F}^n$.

Also recall that $L_A(\mathbb{F}^n) = \mathcal{C}(A)$.

(a) Suppose $\mathbf{p} \in \mathbb{F}^n$ and $\mathbf{b} \in \mathcal{C}(A)$, and suppose '$\mathbf{u} = \mathbf{p}$' is a solution of the linear equation $A\mathbf{u} = \mathbf{b}$ with unknown $\mathbf{u}$ in $\mathbb{F}^n$.

Then ${L_A}^{-1}(\{\mathbf{b}\}) = {L_A}^{-1}(\{L_A(\mathbf{p})\}) = \mathbf{p} + \mathcal{N}(A) = \mathbf{x} + \mathcal{N}(A) = \{\mathbf{y} \in \mathbb{F}^n : \mathbf{y} = \mathbf{p} + \mathbf{h} \text{ for some } \mathbf{h} \in \mathcal{N}(A)\}$.

Recall that $L_A^{-1}(\{\mathbf{b}\})$ is the solution set of the linear equation $A\mathbf{u} = \mathbf{b}$ with unknown $\mathbf{u}$ in $\mathbb{F}^n$.

So the set equality ${L_A}^{-1}(\{\mathbf{b}\}) = \mathbf{p} + \mathcal{N}(A)$ is what we mean by the sentence below:

> *The general solution of the linear equation $A\mathbf{u} = \mathbf{b}$ is given by (any) one particular solution, say '$\mathbf{u} = \mathbf{p}$', of the equation $A\mathbf{u} = \mathbf{b}$ 'added to' the solution space of the homogeneous equation $A\mathbf{u} = \mathbf{0}$.*

(b) The vector space $\mathbb{F}^n/\mathcal{N}(A)$ is isomorphic to $\mathcal{C}(A)$ as vector spaces over $\mathbb{F}$,

A linear isomorphism Υ_{L_A} from $\mathcal{C}(A)$ to $\mathbb{F}^n/\mathcal{N}(A)$ is given by $\Upsilon_{L_A} : \mathbf{b} \longmapsto {L_A}^{-1}(\{\mathbf{b}\})$ for any $\mathbf{b} \in \mathcal{C}(A)$.

This tells us that the set of the non-empty solution sets of the linear equations with the same coefficient matrix A but with various vectors of constant are provided, via Υ_{L_A}, with a natural linear structure, namely, that of $\mathbb{F}^n/\mathcal{N}(A)$.

(c) Suppose $\mathbf{b}, \mathbf{c} \in \mathcal{C}(A)$. Then the equality ${L_A}^{-1}(\{\mathbf{b} + \mathbf{c}\}) = {L_A}^{-1}(\{\mathbf{b}\}) + {L_A}^{-1}(\{\mathbf{c}\})$ holds.

This equality is what we mean by the sentence below:

> *The general solution of the linear equation $A\mathbf{u} = \mathbf{b} + \mathbf{b}$ is the same as 'adding up' the general solution of the equation $A\mathbf{u} = \mathbf{b}$ and the general solution of the equation $A\mathbf{u} = \mathbf{c}$.*

(d) Suppose $\mathbf{b} \in \mathcal{C}(A)$, and $\beta \in \mathbb{F}$. Then the equality ${L_A}^{-1}(\{\beta\mathbf{b}\}) = \beta {L_A}^{-1}(\{\mathbf{b}\})$ holds.

This equality is what we mean by the sentence below:

> *The general solution of the linear equation $A\mathbf{u} = \beta\mathbf{b}$ is the same as 'multiplying' the general solution of the equation $A\mathbf{u} = \mathbf{b}$ by the scalar β.*

21. **Example on quotient spaces: Vector space of indefinite integrals.**

Let J be an open interval in $\mathbb{R}$. Recall that $C(J)$ is the set of all real-valued functions with domain J which is continuous on J. Recall that $C^1(J)$ is the set of all real-valued functions with domain J which is continuously differentiable on J.

Define the function $D : C^1(J) \longrightarrow C(J)$ by $D(h) = h'$ for any $h \in C^1(J)$.

Note that D is a surjective and non-injective linear transformation. (Why?)

Note that the kernel $\mathcal{N}(D)$ of the linear transformation D is the vector space of all constant real-valued functions on J, which is a subspace of $C^1(J)$. (This is a consequence of the Mean-Value Theorem.)

The vector space $C^1(J)/\mathcal{N}(D)$ is isomorphic to $C(J)$ as vector spaces over $\mathbb{R}$.

(a) Suppose $h \in C^1(J)$ and $u \in C(J)$, and suppose $h' = u$.
Then $D^{-1}(\{u\}) = D^{-1}(\{D(h)\}) = h + \mathcal{N}(D) = \{g \in C^1(J) : g = h + C \text{ for some } C \in \mathcal{N}(D)\}$.
Recall that $D^{-1}(\{u\})$ is the set of all primitives of the continuous function u on the interval J. It is the indefinite integral $\int u(x)dx$.
So the set equality $D^{-1}(\{u\}) = h + \mathcal{N}(D)$ is what we mean by the 'formula'

$$\int u(x)dx = h(x) + C, \text{ where } C \text{ is an arbitrary constant}$$

(b) Suppose $u, v \in C(J)$ and $\alpha, \beta \in \mathbb{R}$. Then the equality $D^{-1}(\{\alpha u + \beta v\}) = \alpha D^{-1}(\{u\}) + \beta D^{-1}(\{v\})$ holds. It is what we mean by the 'formula'

$$\int (\alpha u(x) + \beta v(x))dx = \alpha \int u(x)dx + \beta \int v(x)dx.$$