0. (a) The handout is a continuation of the Handout *Linear algebra beyond systems of linear equations and manipulation of matrices.*
   (b) The justification for the theoretical results and the claims in the concrete examples are left as exercises in the use of sets and functions in *set language*.

1. **Definition.**
   Let V be a vector space over a field $\mathbb{F}$, and $\mathbf{x} \in V$.
   (a) *Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k \in V$.*
   $\mathbf{x}$ *is said to be a* **linear combination** *of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ over $\mathbb{F}$ if there exists some $\alpha_1, \alpha_2, \cdots, \alpha_k \in \mathbb{F}$ such that $\mathbf{x} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_k\mathbf{u}_k$.*
   (b) *Let S be a subset of V.*
   $\mathbf{x}$ *is a* **linear combination** *in S over $\mathbb{F}$ if there exist some $k \in \mathbb{N}\backslash\{0\}$, $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_k \in S$ such that $\mathbf{x}$ is a linear combination of $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_k$ over $\mathbb{F}$.*

2. **Definition.**
   Let V be a vector space over a field $\mathbb{F}$. Let S be a subset of V.
   (a) *The* **span of S over $\mathbb{F}$** *is defined to be the set $\{\mathbf{x} \in V : \mathbf{x}$ is a linear combination in S over $\mathbb{F}.\}$. It is denoted by $\mathsf{Span}_{\mathbb{F}}(S)$.*
   (b) *S is called a* **spanning set for V over $\mathbb{F}$** *(or a* **generating set for V over $\mathbb{F}$***) if $V = \mathsf{Span}_{\mathbb{F}}(S)$. (We may also say S* **spans V over $\mathbb{F}$***.)*

3. **Theorem (1).**
   Let V be a vector space over a field $\mathbb{F}$.
   (a) *Suppose S is a subset of V. Then $\mathsf{Span}_{\mathbb{F}}(S)$ is a subspace of V over $\mathbb{F}$.*
   (b) *Suppose U is a subset of V. Then U is a subspace of V over $\mathbb{F}$ iff $\mathsf{Span}_{\mathbb{F}}(U) = U$.*
   (c) *Let S, T be subsets of V. Suppose $S \subset T$. Then $\mathsf{Span}_{\mathbb{F}}(S)$ is a subspace of $\mathsf{Span}_{\mathbb{F}}(T)$ over $\mathbb{F}$.*
   (d) *Let S, T be subsets of V. Suppose $S \subset T$, and $\mathsf{Span}_{\mathbb{F}}(S) = V$. Then $\mathsf{Span}_{\mathbb{F}}(T) = V$.*
   (e) *Suppose S is a subset of V. Then $\mathsf{Span}_{\mathbb{F}}(S) = \mathsf{Span}_{\mathbb{F}}(\mathsf{Span}_{\mathbb{F}}(S))$.*
   (f) *Let S, T be subsets of V. Suppose $S \subset \mathsf{Span}_{\mathbb{F}}(T)$. Then $\mathsf{Span}_{\mathbb{F}}(S)$ is a subspace of $\mathsf{Span}_{\mathbb{F}}(T)$ over $\mathbb{F}$.*

4. **Definition.**
   Let V be a vector space over a field $\mathbb{F}$.
   (a) *Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k \in V$. Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ are pairwise distinct.*
      i. *$\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ are said to be* **linearly dependent over $\mathbb{F}$** *if there exists some $\alpha_1, \alpha_2, \cdots, \alpha_k \in \mathbb{F}$, not all zero, such that $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_k\mathbf{u}_k = \mathbf{0}$.*
      ii. *$\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ are said to be* **linearly independent over $\mathbb{F}$** *if $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ are not* **linearly dependent over $\mathbb{F}$***.*
   (b) *Let S be a subset of V.*
      i. *S is said to be* **linear dependent over $\mathbb{F}$** *if there exist some $k \in \mathbb{N}\backslash\{0\}$ and some $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k \in S$ such that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ are pairwise distinct and $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ are linearly dependent over $\mathbb{F}$.*
      ii. *S is said to be* **linearly independent over $\mathbb{F}$** *if S is not linearly dependent over $\mathbb{F}$. (We may also say S is a* **linear independent set over $\mathbb{F}$***.)*

5. **Lemma (2).**
   Let V be a vector space over a field $\mathbb{F}$.
   (a) *Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k \in V$. Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ are pairwise distinct. Then the statements below are logically equivalent:*
      i. *$\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ are linearly independent over $\mathbb{F}$.*
      ii. *For any $\alpha_1, \alpha_2, \cdots, \alpha_k \in \mathbb{F}$, if $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_k\mathbf{u}_k = \mathbf{0}$ then $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$.*

(b) *Let S be a subset of V. The statements below are logically equivalent:*

i. *S is linearly independent over $\mathbb{F}$.*

ii. *For any $k \in \mathbb{N}\backslash\{0\}$, for any $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k \in S$, if $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ are pairwise distinct and $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_k\mathbf{u}_k = \mathbf{0}$ then $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$.*

Remark. Lemma (2) provides a useful re-formulation for the notion of linear independence, in which the word 'not' has been eliminated.

6. **Theorem (3).**

Let V be a vector space over a field $\mathbb{F}$. The statements below hold:

(a) *For any subsets S, T of V, if S is a subset of T and S is linearly dependent over $\mathbb{F}$, then T is linearly dependent over $\mathbb{F}$.*

(b) *For any subsets S, T of V, if S is a subset of T and T is linearly independent over $\mathbb{F}$, then S is linearly independent over $\mathbb{F}$.*

7. **Definition.**

Let V be a vector space over a field $\mathbb{F}$, and S be a subset of V.

S is said to be a **base for V over $\mathbb{F}$** *if S is a spanning set for V over $\mathbb{F}$ and S is linearly independent over $\mathbb{F}$.*

8. **Theorem (4).**

Let V be a vector space over a field $\mathbb{F}$. The statements below hold:

(a) $\mathsf{Span}_{\mathbb{F}}(\emptyset) = \{\mathbf{0}\}$.

(b) *$\emptyset$ is linear independent over $\mathbb{F}$.*

(c) *$\emptyset$ is a base for the zero subspace $\{\mathbf{0}\}$ of V over $\mathbb{F}$.*

9. **Examples (and non-examples) of spanning sets, linearly independent sets and bases.**

(a) Let $\mathbb{F}$ be a field. Denote by $\mathbf{e}_j^{(n)}$ the column vector whose j-th entry is 1 and whose other entries are all 0.

$\{\mathbf{e}_j^{(n)} \mid j \in [\![1, n]\!]\}$ is a spanning set for $\mathbb{F}^n$ over $\mathbb{F}$. It is a linearly independent set over $\mathbb{F}$. Hence it is a base for $\mathbb{F}^n$.

$\{\mathbf{e}_j^{(n)} \mid j \in [\![1, n]\!]\}$ is called the **standard base for $\mathbb{F}^n$ (over $\mathbb{F}$).**

(b) Let $\mathbb{F}$ be a field. Denote by $E_{i,j}^{m,n}$ the $(m \times n)$-matrix with entries in $\mathbb{F}$ whose (i, j)-th entry is 1 and whose other entries are all 0.

i. $\{E_{i,j}^{m,n} \mid i \in [\![1, m]\!] \text{ and } j \in [\![1, n]\!]\}$ is a spanning set for $\mathsf{Mat}_{m\times n}(\mathbb{F})$ over $\mathbb{F}$. It is a linearly independent set over $\mathbb{F}$. Hence it is a base for $\mathsf{Mat}_{m\times n}(\mathbb{F})$ over $\mathbb{F}$.

ii. Recall that $\mathrm{Sym}_n(\mathbb{F})$ is the of all symmetric $(n \times n)$-square matrices with entries in the field $\mathbb{F}$. It is a vector space over $\mathbb{F}$.

$\left\{ \frac{1}{2}(E_{i,j}^{n,n} + E_{j,i}^{n,n}) \;\middle|\; i, j \in [\![1, n]\!] \text{ and } i \leq j \right\}$ is a spanning set for $\mathrm{Sym}_n(\mathbb{F})$. It is a linearly independent set over $\mathbb{F}$. Hence it is a base for $\mathrm{Sym}_n(\mathbb{F})$ over $\mathbb{F}$.

iii. Recall that $\mathrm{Skew}_n(\mathbb{F})$ is the set of all skew-symmetric $(n \times n)$-square matrices with entries in the field $\mathbb{F}$. It is a vector space over $\mathbb{F}$.

$\left\{ \frac{1}{2}(E_{i,j}^{n,n} - E_{j,i}^{n,n}) \;\middle|\; i, j \in [\![1, n]\!] \text{ and } i < j \right\}$ is a spanning set for $\mathrm{Skew}_n(\mathbb{F})$. It is a linearly independent set over $\mathbb{F}$. Hence it is a base for $\mathrm{Skew}_n(\mathbb{F})$ over $\mathbb{F}$.

(c) Denote by $e_j(x)$ the polynomial x^j for each $j \in \mathbb{N}$. ($e_0(x)$ is the constant polynomial 1.)

i. $\{e_j(x) \mid j \in \mathbb{N}\}$ is a spanning set for the vector space $\mathbb{R}[x]$ over $\mathbb{R}$. It is a linearly independent set over $\mathbb{R}$. Hence it is a base for $\mathbb{R}[x]$ over $\mathbb{R}$.

ii. Recall that for each non-negative integer n, $\mathbb{R}_{\leq n}[x]$ is the set of all polynomials with real coefficients and of degree at most n. It is a vector space over $\mathbb{R}$.

$\{e_j(x) \mid j \in [\![0, n]\!]\}$ is a spanning set for $\mathbb{R}_{\leq n}[x]$ over $\mathbb{R}$. It is a linearly independent set over $\mathbb{R}$. Hence it is a base for $\mathbb{R}_{\leq n}[x]$ over $\mathbb{R}$.

(d) i. Let $n \in \mathbb{N}$. Write $S = [\![0, n]\!]$.

Let $\mathbb{F}$ be a field. Recall that $\mathsf{Map}(S, \mathbb{F})$ is the set of all functions with domain S and range $\mathbb{F}$. It is a vector space over $\mathbb{F}$.

For each $j \in S$, define the function $\delta_j : S \longrightarrow \mathbb{F}$ by $\delta_j(x) = \begin{cases} 1 & \text{if } x = j \\ 0 & \text{if } x \neq j \end{cases}$.

$\{\delta_j \mid j \in S\}$ is a spanning set for $\mathsf{Map}(S, \mathbb{F})$ over $\mathbb{F}$. It is linearly independent over $\mathbb{F}$. Hence it is a base for $\mathsf{Map}(S, \mathbb{F})$ over $\mathbb{F}$.

ii. Let $\mathbb{F}$ be a field. Recall that $\mathsf{Map}(\mathbb{N}, \mathbb{F})$ is the set of all infinite sequences with entries in $\mathbb{F}$. It is a vector space over $\mathbb{F}$.

For each $j \in \mathbb{N}$, define $\delta_j : \mathbb{N} \longrightarrow \mathbb{F}$ by $\delta_j(x) = \begin{cases} 1 & \text{if } x = j \\ 0 & \text{if } x \neq j \end{cases}$.

$\{\delta_j \mid j \in \mathbb{N}\}$ is linearly independent over $\mathbb{F}$. However, it is not a spanning set for $\mathsf{Map}(\mathbb{N}, \mathbb{F})$ over $\mathbb{F}$.

iii. Recall that $\mathsf{Map}(\mathbb{R}, \mathbb{R})$ is the set of all real-valued functions with domain $\mathbb{R}$. It is a vector space over $\mathbb{R}$.

For each $r \in \mathbb{R}$, define the function $\delta_r : \mathbb{R} \longrightarrow \mathbb{R}$ by $\delta_r : \mathbb{R} \longrightarrow \mathbb{R}$ by $\delta_r(x) = \begin{cases} 1 & \text{if } x = r \\ 0 & \text{if } x \neq r \end{cases}$.

$\{\delta_j \mid j \in S\}$ is a linearly independent subset of $\mathsf{Map}(\mathbb{R}, \mathbb{R})$ over $\mathbb{R}$. However, it is not a spanning set for $\mathsf{Map}(\mathbb{R}, \mathbb{R})$ over $\mathbb{R}$.

(e) i. $\mathbb{C}$ is a vector space over $\mathbb{R}$.

$\{1, i\}$ is a spanning set for $\mathbb{C}$ over $\mathbb{R}$. It is linearly independent over $\mathbb{R}$. Hence it is a base for $\mathbb{C}$ over $\mathbb{R}$.

ii. Write $\mathbb{E} = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$.

$\mathbb{E}$ is a field extension of $\mathbb{Q}$. Therefore $\mathbb{E}$ is a vector space over $\mathbb{Q}$.

$\{1, \sqrt{2}\}$ is a spanning set of $\mathbb{E}$ over $\mathbb{Q}$. It is linearly independent over $\mathbb{Q}$. Hence it is a base for $\mathbb{E}$ over $\mathbb{Q}$.

iii. Write $\mathbb{E} = \{a + bi + c\sqrt{2} + di\sqrt{2} \mid a, b, c, d \in \mathbb{Q}\}$.

$\mathbb{E}$ is a field extension of $\mathbb{Q}$. Therefore $\mathbb{E}$ is a vector space over $\mathbb{Q}$.

$\{1, i, \sqrt{2}, \sqrt{2}i\}$ is a spanning set of $\mathbb{E}$ over $\mathbb{Q}$. It is linearly independent over $\mathbb{Q}$. Hence it is a base for $\mathbb{E}$ over $\mathbb{Q}$.

iv. $\mathbb{R}$ is a vector space over $\mathbb{Q}$.

A. $\{1\} \cup \{\sqrt{p} \mid p \text{ is a positive prime number.}\}$ is linearly independent over $\mathbb{Q}$. However, it is not a spanning set for $\mathbb{R}$ over $\mathbb{Q}$.

B. (Take for granted that the number e is transcendental over $\mathbb{Q}$ in the sense that for any polynomial $f(x)$ with rational coefficients, if $f(x)$ is not the zero polynomial then $f(e) \neq 0$.)

$\{e^j \mid j \in \mathbb{Z}\}$ is linearly independent over $\mathbb{Q}$. However, it is not a spanning set for $\mathbb{R}$ over $\mathbb{Q}$.

10. **Theorem (5).**

Let V be a vector space over a field $\mathbb{F}$, and S be a subset of V. The statements below are logically equivalent:

(a) *S is a base for V over $\mathbb{F}$.*

(b) *For any $\mathbf{x} \in V \backslash \{\mathbf{0}\}$, there exist some unique $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k \in S$, $\alpha_1, \alpha_2, \cdots, \alpha_k \in \mathbb{F}$ such that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ are pairwise distinct and $\mathbf{x} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_k\mathbf{u}_k$.*

Remark. Theorem (5) describes the significance of bases for vector spaces: every non-zero vector in a vector space is expressed as a uniquely determined linear combination of vectors in a given base for that vector space. Bases for vector spaces are characterized as such.

11. **Theorem (6).**

Let V be a vector space over a field $\mathbb{F}$. The statements below hold:

(a) *Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k \in V$. Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ are pairwise distinct. The statements below are logically equivalent:*

i. *$\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ are linearly dependent over $\mathbb{F}$.*

ii. *One of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ is a linear combination of the others over $\mathbb{F}$.*

(b) *Let S be a subset of V. Suppose S is linearly dependent over $\mathbb{F}$. Then there exists some $\mathbf{x} \in S$ such that $\mathsf{Span}_{\mathbb{F}}(S) = \mathsf{Span}_{\mathbb{F}}(S \backslash \{\mathbf{x}\})$.*

(c) *Let T be a subset of V. Suppose T is linearly independent over $\mathbb{F}$. Further suppose $\mathsf{Span}_{\mathbb{F}}(T) \subsetneq V$. Then there exists some $\mathbf{y} \in V \backslash \mathsf{Span}_{\mathbb{F}}(T)$ such that $T \cup \{\mathbf{y}\}$ is linearly independent over $\mathbb{F}$.*

Remark. The idea in Statement (b) has been used in your introductory *linear algebra* course. To determine a basis for the column space $\mathcal{C}(A)$ of a given $(m \times n)$-matrix A with, say, real entries, first view $\mathcal{C}(A)$ as the vector space spanned by the set of columns of the matrix A (which are regarded as vectors in $\mathbb{R}^m$). If this set of vectors is not linearly independent over $\mathbb{R}$, then we apply Statement (b) repeatedly, by 'deleting' one vector from S at each step, to obtain 'smaller and smaller' subsets of S which still span thw whole of $\mathcal{C}(A)$, until we obtain a linearly independent subset of S which spans $\mathcal{C}(A)$.

12. **Corollary to Theorem (6).**

Let V be a vector space over a field $\mathbb{F}$, and S be a base for V over $\mathbb{F}$.

Let $\mathbf{y} \in V \backslash \{\mathbf{0}\}$. Suppose $\mathbf{y} \notin S$. Then there exists some $\mathbf{x} \in S$ such that $(S \backslash \{\mathbf{x}\}) \cup \{\mathbf{y}\}$ is a base for V over $\mathbb{F}$.

Remark. This result is known as the **Replacement Theorem**. In plain words, it says that given any given base S for a vector space V over $\mathbb{F}$ and given any non-zero vector $\mathbf{y}$ 'outside' S, we may modify S to obtain another base for V over $\mathbb{F}$ by replacing one appropriate vector in S with the vector y. 'Inductively', for any $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_k \in V \backslash S$, if $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_k$ are pairwise distinct and linearly independent over $\mathbb{F}$, then there exist some $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k \in S$ such that $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k$ are pairwise distinct and $(S \backslash \{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k\}) \cup \{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_k\}$ is a base for V over $\mathbb{F}$.

13. **Lemma (7).**

Let V be a vector space over $\mathbb{F}$, and S, T be subsets of V.

Suppose that S is a subset of T. Further suppose that S is a spanning set for V over $\mathbb{F}$, and T is a linearly independent set over $\mathbb{F}$.

Then $S = T$.

14. **Corollary to Lemma (7).**

Let V be a vector space over $\mathbb{F}$, and S, T be subsets of V.

Suppose that S is a subset of T. Further suppose that each of S, T is a base for V over $\mathbb{F}$.

Then $S = T$.

15. **Theorem (8).**

Let V be a vector space over a field $\mathbb{F}$, and S be a subset of V. The statements below are logically equivalent:

(a) *S is a base for V over $\mathbb{F}$.*

(b) *S is a spanning set for V over $\mathbb{F}$, and the statement* (mSP) *holds:*

(mSP) *For any subset R of V, if R is a subset of S and R is a spanning set for V over $\mathbb{F}$ then $R = S$.*

(c) *S is a linearly independent set over $\mathbb{F}$, and the statement* (MLI) *holds:*

(MLI) *For any subset T of V, if S is a subset of T and T is a linearly independent set over $\mathbb{F}$ then $S = T$.*

Remarks.

- Under the assumption that S is a spanning set for V over $\mathbb{F}$, the statement (mSP) tells us that S is minimal amongst all spanning set for V over $\mathbb{F}$ in the sense that no proper subset of S can be a spanning set for V over $\mathbb{F}$. As a whole, Theorem (8) tells us that S is a base for V over $\mathbb{F}$ iff S is a minimal spanning set for V over $\mathbb{F}$.
- Under the assumption that S is linearly independent over $\mathbb{F}$, the statement (MLI) tells us that S is maximal amongst all linearly independent subsets of V over $\mathbb{F}$ in the sense that no subset of V which contains S as a proper subset can be linearly independent over $\mathbb{F}$. As a whole, Theorem (8) tells us that S is a base for V over $\mathbb{F}$ iff S is a maximal linearly independent subset of V over $\mathbb{F}$.

16. **Definition.**

Let V be a vector space over a field $\mathbb{F}$.

(a) *V is said to be* **finite-dimensional over** $\mathbb{F}$ *if there is a base B for V over $\mathbb{F}$ which is finite set.*

(b) *V is said to be* **infinite-dimensional over** $\mathbb{F}$ *if V is not finite-dimensional over $\mathbb{F}$.*

17. **Lemma (9).**

Let V be a vector space over a field $\mathbb{F}$.

Suppose V is finite-dimensional over $\mathbb{F}$. Then every base for V over $\mathbb{F}$ contains the same number of elements.

Remark. This is a consequence of the Replacement Theorem.

18. **Definition.**

Let V be a finite-dimensional vector space over $\mathbb{F}$. The number of elements in a base for V over $\mathbb{F}$ is called the **dimension of V over $\mathbb{F}$**, *and is denoted by $\dim_{\mathbb{F}}(V)$.*

19. **Examples and non-examples on finite-dimensional vector spaces.**

(a) Let V be a vector space over a field $\mathbb{F}$.
The zero subspace of V over $\mathbb{F}$ is the only zero-dimensional subspace of V over $\mathbb{F}$.

(b) Let $\mathbb{F}$ be a field.
$\mathbb{F}^n$ is a n-dimensional vector space over $\mathbb{F}$.

(c) Let $\mathbb{F}$ be a field.

i. $\mathsf{Mat}_{m\times n}(\mathbb{F})$ over $\mathbb{F}$ is an (mn)-dimensional vector space over $\mathbb{F}$.

ii. $\text{Sym}_n(\mathbb{F})$ is an $\dfrac{n(n+1)}{2}$-dimensional vector space over $\mathbb{F}$.

iii. $\text{Skew}_n(\mathbb{F})$ is an $\dfrac{n(n-1)}{2}$-dimensional vector space over $\mathbb{F}$.

(d) i. $\mathbb{R}[x]$ is an infinite-dimensional vector space over $\mathbb{R}$.

ii. For each non-negative integer n, $\mathbb{R}_{\leq n}[x]$ is an $(n+1)$-dimensional vector space over $\mathbb{R}$.

(e) i. Let $n \in \mathbb{N}$. Write $S = [\![0, n]\!]$. Let $\mathbb{F}$ be a field.
$\mathsf{Map}(S, \mathbb{F})$ is an $(n+1)$-dimensional vector space over $\mathbb{F}$.

ii. $\mathsf{Map}(\mathbb{N}, \mathbb{F})$ is an infinite-dimensional vector space over $\mathbb{F}$.

iii. $\mathsf{Map}(\mathbb{R}, \mathbb{R})$ is an infinite dimensional vector space over $\mathbb{R}$.

iv. Let I be an open interval in $\mathbb{R}$. Recall that $C(I)$ is the set of all real-valued functions of one real variable with domain I which are continuous on I.
$C(I)$ is an infinite-dimensionsal vector space over $\mathbb{R}$.
Recall that for each positive integer n, $C^n(I)$ is the set of all real-valued functions of one real variable with domain I which are n-times continuously differentiable on I.
$C^n(I)$ is an infinite-dimesional vector space over $\mathbb{R}$.

v. Recall that $\ell_2(\mathbb{R})$ is the set of all square-summable infinite sequences of real numbers.
$\ell_2(\mathbb{R})$ is an infinite dimensional vector space over $\mathbb{R}$.

(f) i. $\mathbb{C}$ is a two-dimensional vector space over $\mathbb{R}$.

ii. Write $\mathbb{E} = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$.
$\mathbb{E}$ is a two-dimensional vector space over $\mathbb{Q}$.

iii. Write $\mathbb{E} = \{a + bi + c\sqrt{2} + di\sqrt{2} \mid a, b, c, d \in \mathbb{Q}\}$.
$\mathbb{E}$ is a four-dimensional vector space over $\mathbb{Q}$.

iv. $\mathbb{R}$ is an infinite-dimensional vector space over $\mathbb{Q}$.

20. **Theorem (10).**

Let V be a finite-dimensional vector space over a field $\mathbb{F}$. Write $n = \dim_{\mathbb{F}}(V)$. The statements below hold:

(a) *Suppose U is a subspace of V over $\mathbb{F}$.*
Then U is finite-dimensional over $\mathbb{F}$, and $\dim_{\mathbb{F}}(U) \leq n$. Equality holds iff $W = V$.

(b) *Suppose U is a subspace of V over $\mathbb{F}$. Write $m = \dim_{\mathbb{F}}(U)$, and $k = n - m$. Suppose $k > 0$ and B is a base for U over $\mathbb{F}$.*
Then there exist some $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k \in V \backslash U$ such that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ are pairwise distinct and $B \cup \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k\}$ is a base for V.

(c) *Suppose U is a subspace of V over $\mathbb{F}$. Then there exists some finite-dimensional subspace W over V over $\mathbb{F}$ such that $V = U + W$ and $U \cap W = \{\mathbf{0}\}$. For the same U, W, the equality $\dim_{\mathbb{F}}(V) = \dim_{\mathbb{F}}(U) + \dim_{\mathbb{F}}(W)$ holds.*

Remarks.

- According to Statement (a), V itself is the only subspace of V over $\mathbb{F}$ which is of the same dimension of V over $\mathbb{F}$.
- Statement (c) is an immediately consequence of Statement (a) and Statement (b).

  The proof of Statement (a) and Statement (b) in Theorem (10) relies on the application of Statement (c) in Theorem (6).

21. **How are finite-dimensional vector spaces linked up with matrices?**

Let V be a finite-dimensional vector space over a field $\mathbb{F}$. Write $\dim_{\mathbb{F}}(V) = m$.

Let B be a base for V over $\mathbb{F}$. Denote the m elements of B by $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_m$.

Let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n \in V$, and $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$.

For any $j \in [\![1, n]\!]$, there exist some $a_{1j}, a_{2j}, \cdots, a_{mj} \in \mathbb{F}$ such that $\mathbf{v}_j = a_{1j}\mathbf{u}_1 + a_{2j}\mathbf{u}_2 + \cdots + a_{mj}\mathbf{u}_m$.

For each $j = 1, 2, \cdots, n$, write $\mathbf{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$. Define $A = \left[\begin{array}{c|c|c|c} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{array}\right]$. (Note that A is an $(m \times n)$-matrix with entries in $\mathbb{F}$, whose null space $\mathcal{N}(A)$ is a subspace of $\mathbb{F}^n$ over $\mathbb{F}$ and whose column space $\mathcal{C}(A)$ is a subspace of $\mathbb{F}^m$ over $\mathbb{F}$.)

Let $\mathbf{w} \in V$. There exist some $b_1, b_2, \cdots, b_m \in \mathbb{F}$ such that $\mathbf{w} = b_1\mathbf{u}_1 + b_2\mathbf{u}_2 + \cdots + b_m\mathbf{u}_m$. Write $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$.

Let $c_1, c_2, \cdots, c_n \in \mathbb{F}$. Write $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$.

Under the above assumptions, the statements (†), (‡) are logically equivalent:

(†) $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$.

(‡) $\mathbf{b} = A\mathbf{c}$.

As a consequence, the statements below hold:

(a) $\mathbf{w} \in \mathsf{Span}_{\mathbb{F}}(S)$ iff $\mathbf{b} \in \mathcal{C}(A)$.

(b) Suppose $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are pairwise distinct. Then $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are linearly dependent over $\mathbb{F}$ iff $\mathcal{N}(A)$ contains a non-zero vector in $\mathbb{F}^n$.

(c) Suppose $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are pairwise distinct. Then $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are linearly independent over $\mathbb{F}$ iff $\mathcal{N}(A) = \{\mathbf{0}\}$.

(d) Suppose $m = n$. Then S is a base for V over $\mathbb{F}$ iff A is non-singular (in the sense that $\mathcal{N}(A) = \{\mathbf{0}\}$).

22. **Theorem (11).**

Let V be a vector space over a field $\mathbb{F}$. Suppose V is infinite-dimensional over $\mathbb{F}$. Then the statements below hold:

(a) *V has a base over $\mathbb{F}$.*

(b) *Any two bases for V over $\mathbb{F}$ are of equal cardinality to each other, in the sense that there is a bijective function from one of them to the other.*

Remark. The proof is omitted. The argument relies on the axioms for set theory.