MATH1050 What is the System of all Natural Numbers?

1. In the late nineteenth century, Giuseppe Peano explained in his *Arithmetices Principia, nova methodo exposita* what the system of all natural numbers is by a system of definitions and axioms. From this system of definitions and axioms, all the properties of the natural numbers familiar to us can be deduced. (This is in the spirit as Euclid in his *Elements*, explaining what the (Euclidean) plane is.) Peano used a lot of symbols; below is what he meant when his symbols are translated into words.

   The sign $\mathbf{N}$ *means (the class of all)* **number** *(***positive integer***).*

   The sign 1 *means* **unity***.*

   The sign $a+1$ *means* **the successor of** a*, or* a **plus** 1.

   The sign $=$ *means* **is equal to***. ...*

   AXIOMS.

   1. $1 \in \mathbf{N}$.
   2. *Suppose* $a \in \mathbf{N}$*. Then* $a = a$.
   3. *Suppose* $a, b \in \mathbf{N}$*. Then* $(a = b$ *iff* $b = a)$.
   4. *Suppose* $a, b, c \in \mathbf{N}$*. Then (if* $(a = b$ *and* $b = c)$ *then* $a = c)$.
   5. *Suppose* $(a = b$ *and* $b \in \mathbf{N})$*. Then* $a \in \mathbf{N}$.
   6. *Suppose* $a \in \mathbf{N}$*. Then* $a + 1 \in \mathbf{N}$.
   7. *Suppose* $a, b \in \mathbf{N}$*. Then* $(a = b$ *iff* $a + 1 = b + 1)$.
   8. *Suppose* $a \in \mathbf{N}$*. Then (it is not true that* $a + 1 = 1)$.
   9. *Suppose* k *is a class. Suppose* $(1 \in k$ *and (for any* $x \in \mathbf{N}$*, if* $x \in k$ *then* $x + 1 \in k))$*. Then the class* $\mathbf{N}$ *is contained in* k.

   We usually refer to Items (1), (6), (7), (8), (9) collectively as Peano's Axioms. Peano started with 'one' instead of 'zero' for his natural numbers.

2. Peano's Axioms may be re-stated as below, with 'zero' instead of 'one' being chosen as the 'starting point':

   The set of all natural numbers $\mathbb{N}$ *is a set which possesses the following properties:*

   (P1) $0 \in \mathbb{N}$.

   (P2) *For any* $n \in \mathbb{N}$*, there is some (unique)* $n^+ \in \mathbb{N}$. (n^+ is called the successor of n.)

   (P3) *For any* $m, n \in \mathbb{N}$*, (if* $m^+ = n^+$ *then* $m = n)$.

   (P4) *For any* $n \in \mathbb{N}$*,* $0 \neq n^+$.

   (P5) *Let* S *be a subset of* $\mathbb{N}$*. Suppose* $0 \in S$*. Also suppose that for any* $n \in \mathbb{N}$*, if* $n \in S$ *then* $n^+ \in S$*. Then* $S = \mathbb{N}$. (This is the Principle of Mathematical Induction.)

   With Peano's Axiom, we obtain not only the set of all natural numbers, but also the 'structure' of which we understand as the natural number system: they are the addition, subtraction, multiplication for natural numbers, together with usual ordering of the natural numbers. Note that the Principle of Mathematical Induction is one of the 'fundamental assumptions' in the natural number system. It is with the help of the Principle of Mathematical Induction that addition and multiplication for natural numbers are constructed. Subtraction is defined in terms of addition. The usual ordering of the natural numbers is ingrained in the definition for the notion of 'successor'. The Well-ordering Principle for integers is a consequence of the Principle of Mathematical Induction.

3. Why do we start with 0 instead of 1 here? One reason is that John von Neumann provided a 'model' for the set of all natural numbers, building up everything from the empty set: such a 'model' satisfies all of Peano's Axioms. The idea behind is described below:

   (a) Write $\emptyset = 0$. (The empty set has 'zero' element.)

   Define $1 = \{\emptyset\}$. (This set contains 'exactly' 'one' object as its element.)

   Define $2 = \{\emptyset, \{\emptyset\}\}$. Note that $2 = \{0, 1\}$. (This set contains 'exactly' 'two' objects as its elements. Of course, we have to prove $0 \neq 1$ to justify this claim.)

   Define $3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$. Note that $3 = \{0, 1, 2\}$. (This set contains 'exactly' 'three' objects as its elements. Of course, we have to further prove $1 \neq 2$, $0 \neq 2$ to justify this claim.)

   Define $4 = \{0, 1, 2, 3\}$. Define $5 = \{0, 1, 2, 3, 4\}$. Define $6 = \{0, 1, 2, 3, 4, 5\}$. Et cetera.

   (b) In general, whenever S is a set, define the set S^+ to be the set $S \cup \{S\}$. We call S^+ the successor of S.

   Now $1 = 0^+, 2 = 1^+, 3 = 2^+, 4 = 3^+, 5 = 4^+, 6 = 5^+, \cdots$.

   Note that by definition, $S \subset S^+$ whenever S is a set.

   In fact, according to the respective definitions of $0, 1, 2, 3, 4, 5, 6, \cdots$, we have $0 \subset 1 \subset 2 \subset 3 \subset 4 \subset 5 \subset 6 \subset \cdots$ as sets.

   (c) In general, whenever M is a set, we call M a successor set if the following statements hold:

   (M1) $\emptyset \in M$.

   (M2) For any object S, if $S \in M$ then $S^+ \in M$.

   (The Principle of Mathematical Induction is ingrained in this definition.)

   (d) $\mathbb{N}$ is defined to be the 'smallest' successor set, in the sense that whenever M is a successor set, $\mathbb{N}$ is a subset of M. We can check that the set $\mathbb{N}$ thus defined satisfies Peano's Axioms. The 'subset relation' defines naturally the usual ordering in $\mathbb{N}$.

   (However, does $\{0, 1, 2, 3, 4, 5, \cdots\}$ form a set at all? Or is there any successor set at all? One purpose of *axiomatic set theory* is to formally assure ourselves that it makes sense to talk about such sets.)