MATH1050 Ideas behind the construction of the real number system with rational numbers

1. In the mid-nineteenth century there were several attempts in making sense of the notions of real numbers and the real number system in terms of rational numbers (which mathematicians then felt more comfortable). It was part of the historical development in which mathematicians tried to 'make the calculus (of one variable) rigorous'.

   It was in this process that mathematicians asked this question:

   - '*What is the real number system?*'

   We have provided an answer in the handout *Formalization of the Real Number System as understood in School Maths*.

   Here we inspect the ideas which motivated the various approaches in the construction of the real number system with rational numbers. We take for granted the validity of Statement ($\sharp$):

   ($\sharp$) *Let $\alpha, \beta \in \mathbb{R}$. Suppose $\alpha < \beta$. Then there exists some $r \in \mathbb{Q}$ such that $\alpha < r < \beta$.*

   Statement ($\sharp$) is Corollary (D2) in the handout *Archimedean Principle for the reals*, which can be deduced with the help of the Least-upper-bound Axiom and the Well-ordering Principle for Integers.

2. **Cantor's construction of the real number system.**

   Georg Cantor visualized a real number as the entirety of those infinite sequences of rational numbers which approximate the real number concerned.

   A motivation for his idea is provided in Theorem (IR1).

   Theorem (IR1).

   Let α be an irrational number.

   Let $\{r_n\}_{n=0}^{\infty}$ be a strictly decreasing infinite sequence of rational numbers which converges to 0. The statements below hold:

   (a) *For any $n \in \mathbb{N}$, there exists some $c_n \in \mathbb{Q}$ such that $\alpha - r_n < c_n < \alpha + r_n$.*

   (b) *$\{c_n\}_{n=0}^{\infty}$ is an infinite sequence of rational numbers which converges to α.*

   (c) *For any positive rational number q, there exist some $N \in \mathbb{N}$ such that for any $m, n \in \mathbb{N}$, if $m > N$ and $n > N$ then $|c_m - c_n| < q$.*

   Statement (a) is an immediate consequence of Statement ($\sharp$).

   Statements (b), (c) are consequences of Statement (a) together with the definition for the notion of *limit of sequence*. (For the definition the notion of *limit of sequence*, refer to *Bounded-Monotone Theorem for infinite sequences*.)

   Remark. What the above says, in plain words, is that it is possible to approximate the irrational number α as accurately as we like with infinite sequences of rational numbers which converges to α, such as $\{c_n\}_{n=0}^{\infty}$.

   The idea that irrational numbers can be approximated as accurately as possible by infinite sequences of rational numbers was exploited by Cantor in his approach in constructing the real number system with rational numbers.

   Definition. (Fundamental sequences.)

   (a) *Let $\{c_n\}_{n=0}^{\infty}$ be an infinite sequence of rational numbers. The sequence $\{c_n\}_{n=0}^{\infty}$ is said to be a* **fundamental sequence** *if the statement* (FS) *holds:*

   (FS) *For any positive rational number q, there exist some $N \in \mathbb{N}$ such that for any $m, n \in \mathbb{N}$, if $m > N$ and $n > N$ then $|c_m - c_n| < q$.*

   (b) *Suppose $\{c_n\}_{n=0}^{\infty}$, $\{c'_n\}_{n=0}^{\infty}$ are fundamental sequences. Then we say $\{c_n\}_{n=0}^{\infty}$ is equivalent to $\{c'_n\}_{n=0}^{\infty}$ if the statement* (EQ) *holds:*

   (EQ) *For any positive rational number q, there exists some $N \in \mathbb{N}$ such that for any $n \in \mathbb{N}$, if $n > N$ then $|c_n - c'_n| < q$.*

   Illustrations.

(a) The infinite sequence of rational numbers $\left\{\sum_{k=0}^{n}\frac{1}{k!}\right\}_{n=0}^{\infty}$ is a fundamental sequence.

The infinite sequence of rational numbers $\left\{\left(1+\frac{1}{n}\right)^n\right\}_{n=1}^{\infty}$ is a fundamental sequence.

These two fundamental sequences are equivalent to each other.

These two fundamental sequences, along with all other fundamental sequences which are equivalent to them, are identified as the irrational number e, according to Cantor.

(b) Let $k \in \mathbb{N}\backslash\{0,1\}$ and p be a positive prime number.

Define the infinite sequence $\{c_n\}_{n=0}^{\infty}$ recursively by

$$\begin{cases} c_0 &= p \\ c_{n+1} &= \frac{1}{k}\left[(k-1)c_n + \frac{p}{{c_n}^{k-1}}\right] \quad \text{for any } n \in \mathbb{N} \end{cases}$$

Note that $\{c_n\}_{n=0}^{\infty}$ is an infinite sequence of rational numbers. We can verify that $\{c_n\}_{n=0}^{\infty}$ is a fundamental sequence.

$\{c_n\}_{n=0}^{\infty}$, along with all fundamental sequences which are equivalent to it, is identified as the irrational number $\sqrt[n]{p}$, according to Cantor.

Further remark. The result below is along a similar line of thought to Theorem (IR1):

- **Theorem (IR1').**

  Let α be a irrational number.

  Let $\{r_n\}_{n=0}^{\infty}$ be a strictly decreasing infinite sequence of positive rational numbers which converges to 0. The statements below hold:

  (a) *For any $n \in \mathbb{N}$, there exists some $a_n, b_n \in \mathbb{Q}$ such that $\alpha - r_n < a_n < \alpha - r_{n+1} < \alpha < \alpha + r_{n+1} < b_n < \alpha + r_n$.*

  (b) *$\{a_n\}_{n=0}^{\infty}$ is a strictly increasing infinite sequence of rational numbers which converges to α.*

  (c) *$\{b_n\}_{n=0}^{\infty}$ is a strictly decreasing infinite sequence of rational numbers which converges to α.*

  The infinite sequence of intervals $\{[a_n, b_n]\}_{n=0}^{\infty}$ is something known as a **nested sequence of interval**, in the sense that $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for any $n \in \mathbb{N}$.

  It happens that its generalized intersection $\bigcap_{n=0}^{\infty}[a_n, b_n]$ is simply the singleton $\{\alpha\}$.

  In this sense, we may say that it is possible to approximate the irrational number α as accurately as we like with an infinite sequence of closed and bounded intervals with rational endpoints which 'will eventually shrink to α', for example $\{[a_n, b_n]\}_{n=0}^{\infty}$.

  This point of view is useful in *numerical mathematics.*

3. **Dedekind's construction of the real number system.**

Richard Dedekind visualized irrational real numbers as 'ideal points' filling the gaps between rational numbers. As for rational numbers, they are just themselves.

A motivation for this idea is provided in Theorem (IR2).

Theorem (IR2).

Let α be an irrational number.

Let $A_\alpha = (-\infty, \alpha) \cap \mathbb{Q}$, $B_\alpha = (\alpha, +\infty) \cap \mathbb{Q}$.

The statements below hold:

(a) ($\star$) *For any $s \in A_\alpha$, for any $t \in B_\alpha$, $s < \alpha < t$.*

($\dagger$) $A_\alpha \cap B_\alpha = \emptyset$.

($\ddagger$) $A_\alpha \cup B_\alpha = \mathbb{Q}$.

(b) i. *A_α is bounded above in $\mathbb{R}$ by every element of B_α.*

ii. *A_α has no greatest element.*

(c) i. *B_α is bounded below in $\mathbb{R}$ by every element of A_α.*

ii. *B_α has no least element.*

(d) i. *The supremum of A_α in $\mathbb{R}$ is α.*

ii. *The infimum of B_α in $\mathbb{R}$ is α.*

The proofs for Statements ($\star$), ($\dagger$), ($\ddagger$) are straightforward exercises in set language and inequalities.

Statements (b.i), (c.i) are immediate consequences of Statement ($\star$).

We prove Statements (b.ii), (d.i) below. (The proofs for Statements (c.ii), (d.ii) are similar.)

Proof of Statement (b.ii).

Suppose it were true that A_α had a greatest element, say, s.

By the definition of A_α, $s < \alpha$. Then, by Statement ($\sharp$), there would exist some $s' \in \mathbb{Q}$ such that $s < s' < \alpha$.

By the definition of A_α, we would have $s' \in A_\alpha$. (Now $s' > s$, and s was by assumption a greatest element of A_α.) Contradiction arises.

Hence A_α has no greatest element in the first place.

Proof of Statement (d.i).

By the definition of A_α, α is an upper bound of A_α in $\mathbb{R}$.

We verify that for any $\beta \in \mathbb{R}$, if β is an upper bound of A_α in $\mathbb{R}$, then $\alpha \leq \beta$:

Pick any $\beta \in \mathbb{R}$. Suppose β is an upper bound of A_α in $\mathbb{R}$.

Suppose it were true that $\beta < \alpha$.

By Statement ($\sharp$), there would exist some $s \in \mathbb{Q}$ such that $\beta < s < \alpha$.

Then, by the definition of A_α, we would have $s \in A_\alpha$. (Now $s > \beta$, and β was assumed to be an upper bound of A_α.) Contradiction arises.

It follows that $\alpha \leq \beta$ in the first place.

Hence α is the supremum of A_α in $\mathbb{R}$.

Remark. What the above says, in plain words, is that the irrational number α 'splits' $\mathbb{Q}$ into two 'disjoint' sets of rationals, namely A_α, B_α, every rational in A_α being strictly less than every rational in B_α.

The pair of sets A_α, B_α is called the **Dedekind cut induced by the irrational number** α.

The idea that an arbitrary irrational number corresponds to a Dedekind cut induced by that irrational number was exploited by Dedekind in his approach in constructing the real number system with rational numbers.

Definition. (Dedekind cut.)

Let S, T be non-empty subsets of $\mathbb{Q}$. The pair of sets S, T is called a **Dedekind cut** *if $S \cap T = \emptyset$ and $S \cup T = \mathbb{Q}$ and for any $x \in S$, for any $y \in T$, $x < y$.*

Illustrations.

(a) Define $S = \left\{ x \in \mathbb{Q} : x \leq \sum_{k=0}^{n} \frac{1}{k!} \text{ for some } n \in \mathbb{N} \right\}$, $T = \mathbb{Q} \backslash S$.

The Dedekind cut S, T is identified as the irrational number e, according to Dedekind.

(b) Let p be a positive prime number.

Define $S = \{x \in \mathbb{Q} : x \leq 0 \text{ or } x^n < p\}$, $T = \{x \in \mathbb{Q} : x > 0 \text{ and } x^n > p\}$.

Observe that the sets S, T are defined in terms of rational numbers alone.

The Dedekind cut S, T is identified as the irrational number $\sqrt[n]{p}$, according to Dedekind.

4. **Decimal representation of real numbers.**

Since childhood, we have been used to visualizing real numbers as certain types of infinite series: the (infinite series of) 'decimal representations'. When the process of representations by decimals is clarified, we obtain another construction of the real number system with rational numbers.

We start with a result which is analogous of Division Algorithm for Natural Numbers (Theorem (DAN) in the Handout *Division Algorithm*):

Theorem (DAR).

Let $x, u \in \mathbb{R}$. Suppose $x \geq 0$ and $u > 0$. Then there exist some unique $q \in \mathbb{N}$, $r \in \mathbb{R}$ such that $x = q \times u + r$ and $0 \leq r < u$.

The existence part of Theorem (DAR) relies on the Archimedean Principle and the Well-ordering Principle for Integers. For its proof, imitate how we start the argument for Theorem (D1) in the handout *Archimedean Principle for the reals*.

The argument for the uniqueness part of Theorem (DAR) is almost the same as that for Theorem (DAN) in the handout *Division Algorithm*.

Corollary (DAR1).

Let $x \in \mathbb{R}$. Suppose $x \geq 0$. Then there exist some unique $q \in \mathbb{N}$, $r \in \mathbb{R}$ such that $x = q + r$ and $0 \leq r < 1$.

Remark on terminology. In the context of Corollary (DAR1), We denote the natural number q by $\lfloor x \rfloor$, and call it is called the **integral part of the non-negative real number** x. The number r is referred to as the **non-integral part of the non-negative real number** x.

Definition. (Decimal representation of real numbers between 0 and 1.)

Let $d \in \mathbb{R}$. Suppose $0 \leq d < 1$.

Let $\{d_n\}_{n=0}^{\infty}$ be an infinite sequence in $[\![0, 9]\!]$.

Suppose the infinite sequence $\left\{ \sum_{k=0}^{p} \frac{d_k}{10^{k+1}} \right\}_{p=0}^{\infty}$ converges to d.

Then we say $\left\{ \sum_{k=0}^{p} \frac{d_k}{10^{k+1}} \right\}_{p=0}^{\infty}$ is a **decimal representation of** d. *As a convention, we write $d = 0.d_0d_1d_2d_3d_4\cdots$.*

Remarks.

(I) What we actually mean by '$d = 0.d_0d_1d_2d_3d_4\cdots$' is '$d = \lim_{p\to\infty} \sum_{k=0}^{p} \frac{d_k}{10^{k+1}}$'. So, for instance, when we write '$\frac{1}{3} = 0.\underbrace{333333\cdots}_{\text{all 3's}}$', what we are actually saying is that $\frac{1}{3}$ is the limit of the infinite sequence $\left\{ \sum_{k=0}^{p} \frac{3}{10^{k+1}} \right\}_{p=0}^{\infty}$.

(II) Some real numbers may admit distinct decimal representations.

For example, $\frac{1}{2} = 0.5\underbrace{000000\cdots}_{\text{all 0's}}$ and $\frac{1}{2} = 0.4\underbrace{999999\cdots}_{\text{all 9's}}$. But this is natural in light of the definition of decimal representation in terms of convergence of infinite sequences.

That every real number between 0 and 1 admits a decimal representation is guaranteed by Theorem (DR).

Theorem (DR).

Let $a \in \mathbb{R}$. Suppose $0 \leq a < 1$.

(a) *For any $n \in \mathbb{N}$, define $\widetilde{a_n} = \lfloor 10^{n+1} a \rfloor$.*

$\{\widetilde{a_n}\}_{n=0}^{\infty}$ is an infinite sequence in $\mathbb{N}$.

Moreover, $\left\{ \frac{\widetilde{a_n}}{10^{n+1}} \right\}_{n=0}^{\infty}$ is an increasing infinite sequence of real numbers and converges to a.

(b) *Further define $a_0 = \widetilde{a_0}$. For any $m \in \mathbb{N}\backslash\{0\}$, further recursively define $a_m = 10\widetilde{a_{m-1}} - \widetilde{a_m}$.*

$\{a_m\}_{m=0}^{\infty}$ is an infinite sequence in $[\![0, 9]\!]$.

The infinite sequence $\left\{ \sum_{k=0}^{p} \frac{a_k}{10^{k+1}} \right\}_{p=0}^{\infty}$ is the same as $\left\{ \frac{\widetilde{a_n}}{10^{n+1}} \right\}_{n=0}^{\infty}$. It is a decimal representation of a.

The justification for the convergence of $\left\{\dfrac{\widetilde{a_n}}{10^{n+1}}\right\}_{n=0}^{\infty}$ to a relies on the formal definition for the notion of *limit of sequence*. The rest of the argument for Theorem (DR) is straightforward.

Illustrations of the ideas in Theorem (DR).

(a) Let $a = \dfrac{1}{3}$.

$\widetilde{a_0} = \left\lfloor \dfrac{10}{3} \right\rfloor = 3$, $\widetilde{a_1} = \left\lfloor \dfrac{100}{3} \right\rfloor = 33$, $\widetilde{a_2} = \left\lfloor \dfrac{1000}{3} \right\rfloor = 333$, et cetera. For each $n \in \mathbb{N}$, $a_n = 3$.

A decimal representation for a is $\left\{\displaystyle\sum_{j=0}^{n} \frac{3}{10^{j+1}}\right\}_{n=0}^{\infty}$, as expected.

(b) Let $a = \dfrac{1}{5}$.

$\widetilde{a_0} = \left\lfloor \dfrac{10}{5} \right\rfloor = 2$, $\widetilde{a_1} = \left\lfloor \dfrac{100}{5} \right\rfloor = 20$, $\widetilde{a_2} = \left\lfloor \dfrac{1000}{2} \right\rfloor = 200$, et cetera. We have $a_0 = 2$. For each $n \in \mathbb{N}\backslash\{0\}$, $a_n = 0$.

A decimal representation for a is $\left\{\dfrac{2}{10} + \displaystyle\sum_{j=1}^{n} \frac{0}{10^{j+1}}\right\}_{n=0}^{\infty}$, as expected.

In the light of Theorem (DAR) and Theorem (DR), we may express each non-negative real number x as $x = N.a_0a_1a_2a_3a_4\cdots$, in which N is the integral part of x, and $0.a_0a_1a_2a_3a_4\cdots$ is a decimal representation of the non-integral part of x. We refer to $N.a_0a_1a_2a_3a_4\cdots$ as a **decimal representation of the non-negative real number** x.

When y is a negative real number, $-y$ is a positive real number, and admits a decimal representation $-y = M.b_0b_1b_2b_3b_4\cdots$. We may express y as $y = -M.b_0b_1b_2b_3b_4\cdots$. We refer to $-M.b_0b_1b_2b_3b_4\cdots$ as a **decimal representation of the negative real number** y.