MATH1050 Surds of positive real numbers.

0. Refer to the handout *Formalization of the Real Number System as understood in School Maths*. There we have introduced the Laws of Arithmetic for the reals and the Laws of Order for the reals, which cover the properties of the arithmetic operations $+, -, \times, \div$ and the symbols $\leq, <$ in the 'world of real numbers'.

   But how about the 'operation' of 'extracting the n-th root for an arbitrary positive real number' (which we seem to have learnt a lot about since school days)?

   It turns out that this 'operation' is much more complicated than we realized in the past. It cannot be described by $+, -, \times, \div$ and $\leq, <$ alone. 'Taking limit' is involved.

1. **Definition. (Real n-th root of a positive real number.)**

   Let n be an integer greater than 1, and r, ρ be positive real numbers. We say ρ is the real n-th root of x if $\rho^n = r$.

   We write $\rho = \sqrt[n]{r}$.

   Remark. For this definition to make sense, Theorem (1) and Theorem (2) need to be proved. These two results are logically independent of each other.

2. **Theorem (1). (Existence of a real n-th root of a positive real number.)**

   Let n be an integer greater than 1. Suppose r is a positive real number. Then there exists some positive real number ρ such that $\rho^n = r$.

   Remark. This result is non-trivial. The argument relies on what you have learnt in the *calculus of one real variable*.

   Theorem (2). (Uniqueness of a real n-th root of a positive real number.)

   Let n be an integer greater than 1. Suppose r is a positive real number. Then there is at most one positive real number ρ which satisfies $\rho^n = r$.

   Remark. A more formal formulation of Theorem (2) reads:

   > *Let n be an integer greater than 1. Suppose r is a positive real number. Suppose ρ, σ are positive real numbers, and $\rho^n = r$ and $\sigma^n = r$. Then $\rho = \sigma$.*

3. **Proof of Theorem (2).**

   Let n be an integer greater than 1. Suppose r is a positive real number. Suppose ρ, σ are positive real numbers, and $\rho^n = r$ and $\sigma^n = r$.

   Then $\rho^n = r = \sigma^n$. Therefore

   $$\begin{aligned} 0 &= \rho^n - \sigma^n \\ &= (\rho - \sigma)(\rho^{n-1} + \rho^{n-2}\sigma + \rho^{n-3}\sigma^2 + \cdots + \rho^{n-k}\sigma^k + \cdots + \rho^2\sigma^{n-3} + \rho\sigma^{n-2} + \sigma^{n-1}) \end{aligned}$$

   Since ρ, σ are positive real numbers, each of ρ^{n-1}, $\rho^{n-2}\sigma$, $\rho^{n-3}\sigma^2$, ..., $\rho^2\sigma^{n-3}$, $\rho\sigma^{n-2}$, σ^{n-1} is a positive real number.

   Then $(\rho^{n-1} + \rho^{n-2}\sigma + \rho^{n-3}\sigma^2 + \cdots + \rho^{n-k}\sigma^k + \cdots + \rho^2\sigma^{n-3} + \rho\sigma^{n-2} + \sigma^{n-1})$ is a positive real number.

   Therefore $\rho - \sigma = 0$. Hence $\rho = \sigma$.

4. Recall the Intermediate-Value Theorem from your *calculus of one real variable* course.

   Intermediate-Value Theorem. (Bolzano's version.)

   Let $a, b \in \mathbb{R}$, with $a < b$. Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function. Suppose $f(a)f(b) < 0$. Suppose f is continuous on $[a, b]$. Then there exists some $x_0 \in (a, b)$ such that $f(x_0) = 0$.

   With the help of the Intermediate-Value Theorem, we can give a proof of Theorem (1).

5. **Proof of Theorem (1), with the help of the Intermediate-Value Theorem.**

   Let n be an integer greater than 1. Suppose r is a positive real number.

   Let $f : [0, r + 1] \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x^n - r$ for any $x \in [0, r + 1]$.

   Note that f is continuous on $[0, r + 1]$ (because f is a polynomial function).

   Also note that $f(0) = -r < 0$.

   We verify that $f(r + 1) > 0$:

Note that $f(r+1) = (r+1)^n - r = (r+1)^n - (r+1) + 1 = (r+1)[(r+1)^{n-1} - 1] + 1$.

Since $r > 0$, we have $r + 1 > 1$. Then $(r+1)^{n-1} > 1$.

Therefore $f(r+1) > 0$.

Now we have $f(0) < 0$ and $f(r+1) > 0$. Then $f(0)f(r+1) < 0$.

By the Intermediate-Value Theorem, there exists some $\rho \in (0, r+1)$ such that $f(\rho) = 0$.

By definition, ρ is a positive real number. Moreover, $\rho^n - r = f(\rho) = 0$. Then $\rho^n = r$.

6. The Intermediate-Value Theorem is non-trivial. It relies on the Bounded-Monotone Theorem, which can be regarded as a 'fundamental assumption' about the real number system.

   Bounded-Monotone Theorem for infinite sequences of real numbers.

   Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence of real numbers.

   Suppose $\{a_n\}_{n=0}^{\infty}$ is $\left\{ \begin{array}{c} \textit{increasing} \\ \textit{decreasing} \end{array} \right\}$.

   Further suppose $\{a_n\}_{n=0}^{\infty}$ is $\left\{ \begin{array}{c} \textit{bounded above} \\ \textit{bounded below} \end{array} \right\}$ in $\mathbb{R}$.

   Then $\{a_n\}_{n=0}^{\infty}$ converges in $\mathbb{R}$.

   We are going to give an outline of an alternative argument for Theorem (1), which relies on the Bounded-Monotone Theorem at the crucial step.

7. **Outline of an alternative proof of Theorem (1), with a direct application of the Bounded-Monotone Theorem.**

   Let n be an integer greater than 1. Suppose r is a positive real number.

   Define the infinite sequence of positive real numbers $\{a_k\}_{k=0}^{\infty}$ by

$$\left\{ \begin{array}{lcl} a_0 & = & r \\ a_{k+1} & = & \dfrac{1}{n}\left[(n-1)a_k + \dfrac{r}{{a_k}^{n-1}}\right] \quad \text{if} \quad k \in \mathbb{N} \end{array} \right. .$$

   We verify the statements below:

   (a) $\{a_k\}_{k=0}^{\infty}$ is strictly decreasing.

   (b) $\{a_k\}_{k=0}^{\infty}$ is bounded below in $\mathbb{R}$.

   By the Bounded-Monotone Theorem, $\{a_k\}_{k=0}^{\infty}$ converges to some real number, which we denote by ρ.

   We have $\lim_{k\to\infty} a_k = \rho$ and $\lim_{k\to\infty} a_{k+1} = \rho$.

   By definition, we have $n{a_k}^{n-1}a_{k+1} = (n-1){a_k}^n + r$ for any $k \in \mathbb{N}$. Then

$$n\rho^n = \cdots = \lim_{k\to\infty} n{a_k}^{n-1}a_{k+1} = \lim_{k\to\infty} [(n-1){a_k}^n + r] = \cdots = (n-1)\rho^n + r.$$

   Therefore $\rho^n = r$.