MATH1050 Formalization of the Real Number System as understood in School Maths

1. In school maths we take the real number system and everything about it for granted. But you will soon find yourself in a mist if you ask questions like:—

   - '*What is the real number system?*'
   - '*Why are the various properties of the real number system as stated in school maths textbooks valid?*'

   A standard 'school-maths textbook' treatment on the real number system probably proceeds as described below:—

   (a) The reader will most likely be told that real numbers are exactly the terminating or non-terminating decimals.

   Some examples of numbers (such as $0, 1, 2, 2.5, 2.\dot{7}\dot{2}, 22/7, \pi, -4$) are presented as examples of 'real numbers'.

   The 'collection of all real numbers' are provided the imagery of an 'infinite line' (in the 'infinite plane'). When identified as the 'collection of all real numbers', this line is called the 'real line'.

   The question why points (which are geometric objects) can be identified as terminating or non-terminating decimals is hushed.

   (b) Next the reader is convinced (or be made to believe) that it makes sense to talk about addition, subtraction, multiplication, division for real numbers, and it also makes sense to compare values of real numbers.

   (c) Certain rules governing the use of the arithmetic operations $+$, $-$, $\times$, $\div$ and the 'ordering symbols' $\leq$, $<$ are introduced, and the reader is convinced (or is made to believe) their validity, 'because' they seem to be valid when they are 'restricted' to the seemingly more familiar types of numbers, such as integers, or rational numbers.

   These rules are left un-proved (or un-justified) in school maths.

   (d) To summarize, everything about the real number system is treated as 'facts', tacitly assumed to be valid.

   This is ironical: these 'facts' about the real number system, which are extensively used for the justification of everything else in mathematics, are left unexplained.

   Such a lack of appropriate treatment is expected, for a simple reason. To be in a position to justify something, some basis of justification must be agreed upon in the first place. Such a basis is made up of:—

   - appropriate definitions for the objects under question, and
   - some statements concerned with those objects under question which are assumed to be true without proof.

   However, in school maths, real numbers and $+$, $-$, $\times$, $\div$, $\leq$, $<$ have been left undefined in the first place.

   This lands ourselves onto a serious problem:

   ($\sharp$) on what (philosophical) grounds should we expect ourselves to be rigorous on various matters which involves the real number system if we are not sure the real number system is 'safe' (or 'safely true') mathematically?

2. There are two possible ways to tackle the problem ($\sharp$):

   - *Method 1.* Construct the real number system out of something which (we believe) is fundamental enough, and then prove everything about the real number system in terms of that *something* which is fundamental enough.
   - *Method 2.* Formalize the real number system by setting up a system of axioms which governs everything concerned with it, and stick to such a system throughout.

   We can make the real number system 'safe' mathematically by either method.

   Method 1 (the 'constructive approach') is protracted and unsavoury.

   Method 2 (the 'axiomatic approach') looks un-natural, especially when un-motivated.

   In any case, at this moment, we do not have at our disposal enough mathematical tools for either method to be applied.

   We are going to take a preparatory step that will pave the way to a solution for the problem ($\sharp$) (through either method): we formalize the standard 'school-maths textbook' treatment on the real number system.

3. We take for granted that it makes sense to talk about objects known as real numbers, 'operations' known as addition, multiplication for real numbers, and 'non-negativity' for real numbers.

   We also select a small amount of statements, formulated in terms of addition, multiplication and non-negativity alone, whose validity will be taken for granted.

   In these things we adhere our school maths understanding on the real number system.

   However, we shall make sure whatever else about the real number system will be defined and/or justified (as carefully as possible) in terms of the above.

   If, and when, we apply Method 1 to make sense of the real number system, the selected statements will be the ones that needs to be justified from what is regarded to be more fundamental.

   If, and when, we apply Method 2 to make sense of the real number system, the selected statements will serve as the 'axioms' for the real number system.

4. We will agree that the **real number system**, as understood in school maths, consists of:—

   - a set, denoted by $\mathbb{R}$, whose elements are called **real numbers**, amongst them two distinct real numbers called **zero**, **one** and denoted by 0, 1 respectively,
   - the arithmetic operations $+$, $\times$, called **addition, multiplication in the reals** respectively) , and
   - a subset of $\mathbb{R}$, denoted by $\mathbb{R}_{\geq 0}$, whose elements are called **non-negative real numbers**,

   forming an **ordered field**, in the sense that the **Laws of Arithmetic for the reals**, namely the statements (A1)-(A11), and the **Laws of Order for the reals (compatible to the arithmetic operations)**, namely the statements (O1)-(O3), are true statements.

5. **Warning.**

   What has been stated above about the real number system does not cover the crucial property known as **analytic completeness**, which will be required to make sense of doing infinitesimal calculus in the real number system.

   In fact, without the help of analytic completeness, we can't even make sense of 'taking square root for arbitrary non-negative real numbers'.

6. **Laws of Arithmetic for the reals.**

   (A1) *For any* $a, b \in \mathbb{R}$, $a + b \in \mathbb{R}$.

   (A2) *For any* $a, b, c \in \mathbb{R}$, $(a + b) + c = a + (b + c)$.

   (A3) *There exists some* $z \in \mathbb{R}$, *namely* $z = 0$, *such that for any* $a \in \mathbb{R}$, $a + z = a$ *and* $z + a = a$.

   (A4) *For any* $a \in \mathbb{R}$, *there exists some* $b \in \mathbb{R}$, *called an* **additive inverse of** a, *such that* $a + b = 0$ *and* $b + a = 0$.

   (A5) *For any* $a, b \in \mathbb{R}$, $a + b = b + a$.

   (A6) *For any* $a, b \in \mathbb{R}$, $a \times b \in \mathbb{R}$.

   (A7) *For any* $a, b, c \in \mathbb{R}$, $(a \times b) \times c = a \times (b \times c)$.

   (A8) *There exists some* $u \in \mathbb{R}$, *namely* $u = 1$, *such that for any* $a \in \mathbb{R}$, $a \times u = a$ *and* $u \times a = a$.

   (A9) *For any* $a \in \mathbb{R}\backslash\{0\}$, *there exists some* $b \in \mathbb{R}$, *called a* **multiplicative inverse of** a, *such that* $a \times b = 1$ *and* $b \times a = 1$.

   (A10) *For any* $a, b \in \mathbb{R}$, $a \times b = b \times a$.

   (A11) *For any* $a, b, c \in \mathbb{R}$, $(a + b) \times c = (a \times c) + (b \times c)$ *and* $a \times (b + c) = (a \times b) + (a \times c)$.

   We can make sure whatever else about addition, subtraction, multiplication and division for the real number system can be defined and/or justified in terms (A1)-(A11) alone, as long as inequality is not involved.

   One implication of this is that almost everything you learn about matrices and vectors with real entries in your beginning *linear algebra* course can be traced back to this point.

   In this handout we are going to provide only a few samples which bring out the flavour.

7. **Some basic statements about $\mathbb{R}$ deduced from the Laws of Arithmetic for the reals alone.**

   (a) **Statement (1a).**
   For any $z' \in \mathbb{R}$, if (for any $a \in \mathbb{R}$, $a + z' = a$ and $z' + a = a$) then $z' = 0$.
   (In this sense 0 is unique additive identity in $\mathbb{R}$.)

   (b) **Statement (1b).**
   For any $u' \in \mathbb{R}$, if (for any $a \in \mathbb{R}$, $a \times u' = a$ and $u' \times a = a$) then $u' = 1$.
   (In this sense 1 is the unique multiplicative identity in $\mathbb{R}$.)

   (c) **Statement (1c).**
   For any $a, b, c \in \mathbb{R}$, if $a + b = 0 = b + a$ and $a + c = 0 = c + a$ then $b = c$.
   (By virtue of (A4) and Statement (1c), the additive inverse of each real number a is unique, and we denote it by $-a$.)

   (d) **Statement (1d).**
   For any $a, b, c \in \mathbb{R}$, if $a \neq 0$ and $a \times b = 1 = b \times a$ and $a \times c = 1 = c \times a$ then $b = c$.
   (By virtue of (A9) and Statement (1d), the multiplicative inverse of each non-zero real number a is unique, and we denote it by a^{-1}.)

8. We give the proof of Statement (1a) below. The proofs of Statements (1b), (1c), (1d) are similar.

   Proof of Statement (1a).

   Let $z' \in \mathbb{R}$. Suppose that for any $a \in \mathbb{R}$, $a + z' = a$ and $z' + a = a$.

   Then, in particular, $0 + z' = 0$.

   By (A3), we have $0 + z' = z'$.

   Then $z' = 0 + z' = 0$.

9. **Subtraction '$-$' and division $\div$ in the reals** are defined in terms of addition and multiplication in the reals, under the assumption of the validity of the statements (A1)-(A11):

   - *For any $c, d \in \mathbb{R}$, we define $c - d$ to be the real number $c + (-d)$.*
   - *For any $c \in \mathbb{R}$, for any $d \in \mathbb{R}\backslash\{0\}$, we define $c \div d$ to be the real number $c \times (d^{-1})$.*

10. **Further statements about $\mathbb{R}$ deduced from the Laws of Arithmetic for the reals alone.**

   (a) **Statement (2a).**
   For any $a \in \mathbb{R}$, $a \times 0 = 0$.
   Proof of Statement (2a).
   Let $a \in \mathbb{R}$.
   By (A3), we have $0 + 0 = 0$.
   By (A11), we have $(a \times 0) + (a \times 0) = a \times (0 + 0) = a \times 0$.
   By (A4), there exists some $c \in \mathbb{R}$ such that $(a \times 0) + c = 0$.
   Then we have

   $$a \times 0 = (a \times 0) + 0 = (a \times 0) + [(a \times 0) + c] = [(a \times 0) + (a \times 0)] + c = (a \times 0) + c = 0.$$

   (The first and third equality are due to (A3), (A2) respectively.)

   (b) **Statement (2b).**
   For any $a \in \mathbb{R}$, $-(-a) = a$.
   Proof of Statement (2b).
   Let $a \in \mathbb{R}$. Write $b = -a$.
   By (A4), $a + (-a) = 0$.
   Also by (A4), $(-b) + b = 0$.

Then $a + (-a) = 0 = (-b) + b = [-(-a)] + (-a)$.

Therefore

$$a = a+0 = a+[(-a)+a] = [a+(-a)]+a = \{[-(-a)]+(-a)\}+a = [-(-a)]+[(-a)+a] = [-(-a)]+0 = -(-a).$$

(The first and seventh equalities are due to (A3). The second and sixth equalities are due to (A4). The third and fifth equalities are due to (A2).)

(c) **Statement (2c).**

For any $a, b \in \mathbb{R}$, $a \times (-b) = (-a) \times b = -(a \times b)$, and $(-a) \times (-b) = a \times b$.

Proof of Statement (2c).

Let $a, b \in \mathbb{R}$.

- By (A4), $b + (-b) = 0$.
  Then $(a \times b) + [a \times (-b)] = a \times [b + (-b)] = a \times 0 = 0$.
  (The first equality is due to (A11). The third equality is due to Statement (2a).)
  Then we have

  $$a\times(-b) = 0+[a\times(-b)] = \{[-(a\times b)]+(a\times b)\}+[a\times(-b)] = [-(a\times b)]+\{(a\times b)+[a\times(-b)]\} = [-(a\times b)]+0 = -(a\times b).$$

  (The first and fifth equalities are due to (A3). The second equality is due to (A4). The third equality is due to (A2).)
- By (A5), $(-a) \times b = b \times (-a)$.
  We have $b \times (-a) = -(b \times a)$.
  Again by (A5), $b \times a = a \times b$.
  Therefore $(-a) \times b = b \times (-a) = -(b \times a) = -(a \times b)$.
- We have $(-a) \times (-b) = -[a \times (-b)] = -[-(a \times b)] = a \times b$.

(d) **Statement (2d).**

For any $a, b \in \mathbb{R}$, $-(a + b) = (-a) + (-b)$.

Proof of Statement (2d). Exercise.

(e) **Statement (2e).**

For any $a, b \in \mathbb{R}$, if $a \times b = 0$ then $a = 0$ or $b = 0$.

(In this sense we say that there is no zero-divisor in $\mathbb{R}$.)

Proof of Statement (2e).

Let $a, b \in \mathbb{R}$. Suppose $a \times b = 0$.

Note that $a = 0$ or $a \neq 0$.

- (Case 1). Suppose $a = 0$. Then $a = 0$ or $b = 0$.
- (Case 2). Suppose $a \neq 0$. By (A9), there exists some $c \in \mathbb{R}$ such that $c \times a = 1$.

Then we have

$$b = 1 \times b = (c \times a) \times b = c \times (a \times b) = c \times 0 = 0.$$

(The first and third equalities are due to (A8), (A7) respectively.)

(f) **Statement (2f).**

For any $a, b, c \in \mathbb{R}$, if $a \neq 0$ and $a \times b = a \times c$ then $b = c$.

Proof of Statement (2f).

Let $a, b, c \in \mathbb{R}$. Suppose $a \neq 0$ and $a \times b = a \times c$.

By (A10), there exists some $d \in \mathbb{R}$ such that $d \times a = 1$.

Then

$$b = 1 \times b = (d \times a) \times b = d \times (a \times b) = d \times (a \times c) = (d \times a) \times c = 1 \times c = c.$$

(The first and seventh equalities are due to (A8). The third and fifth equalities are due to (A7).)

11. **Laws of Order for the reals, compatible with the Laws of Arithmetic.**

(O1) *For any* $a, b \in \mathbb{R}_{\geq 0}$, $a + b \in \mathbb{R}_{\geq 0}$ *and* $a \times b \in \mathbb{R}_{\geq 0}$.

(O2) *For any* $a \in \mathbb{R}$, $a \in \mathbb{R}_{\geq 0}$ *or* $-a \in \mathbb{R}_{\geq 0}$.

(O3) *For any* $a \in \mathbb{R}$, *if* $a \in \mathbb{R}_{\geq 0}$ *and* $-a \in \mathbb{R}_{\geq 0}$ *then* $a = 0$.

We can make sure whatever else about addition, subtraction, multiplication, division and inequalities for the real number system can be defined and/or justified in terms of (A1)-A(11) and (O1)-(O3) alone, as long as 'infinite processes' are not involved.

12. The **usual ordering for the reals**, which is denoted by $\leq$, is defined in terms of subtraction and non-negative real numbers:

   - *For any* $a, b \in \mathbb{R}$, *we say* a *is less than or equal to* b, *and write* $a \leq b$, *if* $b - a \in \mathbb{R}_{\geq 0}$.

   For any real numbers a, b, we agree to write '$a \leq b$' also as '$b \geq a$'. Furthermore, for any real numbers c, d, if $c \leq d$ and $c \neq d$, we agree to write '$c < d$', or equivalently '$d > c$'.

   For each real number b, we say that it is **positive** (or **negative** respectively) if $b > 0$ (or $b < 0$ respectively). For each real number c, we say that it is **non-positive** if $c \leq 0$.

   With the symbol '$\leq$', we can re-write the statements (O1)-(O3) as:

(O1) *For any* $a, b \in \mathbb{R}$, *if* $a \geq 0$ *and* $b \geq 0$ *then* $a + b \geq 0$ *and* $a \times b \geq 0$.

(O2) *For any* $a \in \mathbb{R}$, $a \geq 0$ *or* $-a \geq 0$.

(O3) *For any* $a \in \mathbb{R}$, *if* $a \geq 0$ *and* $-a \geq 0$ *then* $a = 0$.

13. By virtue of the validity of the statements (A1)-(A11) and (O1)-(O3), we can deduce the statements (UO1)-(UO4):

(UO1) *For any* $a \in \mathbb{R}$, $a \leq a$.

(UO2) *For any* $a, b \in \mathbb{R}$, *if* $a \leq b$ *and* $b \leq a$ *then* $a = b$.

(UO3) *For any* $a, b, c \in \mathbb{R}$, *if* $a \leq b$ *and* $b \leq c$ *then* $a \leq c$.

(UO4) *For any* $a \in \mathbb{R}$, $a \leq 0$ *or* $0 \leq a$.

   By virtue of the validity of the statements (UO1)-(UO4), $\leq$ defines a total ordering in $\mathbb{R}$.

   Statements (UO1)-(UO4) are often collectively presented as Statement (UO1)-(UO3) together with Statement (UO5):

(UO5) *For any* $a \in \mathbb{R}$, *exactly one of* '$a < 0$', '$a = 0$', '$a > 0$' *is true.*

   (UO5) is called the **Law of Trichotomy for the reals**.

14. **Absolute value for real numbers** is defined in terms of $\leq$ as well:

   - For any $a \in \mathbb{R}$, we define the absolute value $|a|$ of the number a by

$$|a| = \begin{cases} a & \text{if} \quad a \geq 0 \\ -a & \text{if} \quad a < 0 \end{cases}$$

15. **Some basic statements about inequalities for real numbers deduced from the Laws of Arithmetic for the reals and the Laws of Order for the reals.**

   (a) **Statement (3a).**
   For any $a \in \mathbb{R}$, $a^2 \geq 0$.
   ('a^2' is understood as a short-hand for '$a \times a$'.)
   Proof of Statement (3a).
   Let $a \in \mathbb{R}$.
   By (O2), $a \geq 0$ or $-a \geq 0$.

- (Case 1). Suppose $a \geq 0$. Then by (O1), $a^2 = a \times a \geq 0$.
- (Case 2). Suppose $-a \geq 0$. Then

$$a^2 = a \times a = (-a) \times (-a) \geq 0.$$

(The second equality is due to Statement (2b). The inequality is due to (O1).)

Therefore, in any case, $a^2 \geq 0$.

(b) **Statement (3b).**

For any $a, b \in \mathbb{R}$, if $a \geq 0$ and $b \leq 0$ then $a \times b \leq 0$.

Proof of Statement (3b).

Let $a, b \in \mathbb{R}$. Suppose $a \geq 0$ and $b \leq 0$.

By definition, since $b \leq 0$, we have $-b \geq 0$.

Then $a \geq 0$ and $-b \geq 0$.

By (O1), $a \times (-b) \geq 0$.

By Statement (2c), $a \times (-b) = -(a \times b)$.

Then $-(a \times b) \geq 0$.

Therefore, by definition, $a \times b \leq 0$.

(c) **Statement (3c).**

For any $a, b, c \in \mathbb{R}$, if $a \leq b$ then $a + c \leq b + c$.

Proof of Statement (3c).

Let $a, b, c \in \mathbb{R}$. Suppose $a \leq b$.

Then, by definition, $b - a \geq 0$.

Note that

$$\begin{aligned}
(b+c) - (a+c) &= (b+c) + [-(a+c)] \\
&= (b+c) + [(-a) + (-c)] \\
&= b + \{c + [(-a) + (-c)]\} \\
&= b + \{c + [(-a) + (-c)]\} \\
&= b + \{c + [(-c) + (-a)]\} \\
&= b + \{[c + (-c)] + (-a)\} = b + [0 + (-a)] \\
&= b + (-a) \\
&= b - a.
\end{aligned}$$

(Fill in the reason for each equality.)

Since $b - a \geq 0$, we have $(b+c) - (a+c) \geq 0$ also.

Then $a + c \leq b + c$ by definition.

(d) **Statement (3d).**

For any $a, b, c, d \in \mathbb{R}$, if $a \leq b$ and $c \leq d$ then $a + c \leq b + d$.

Proof of Statement (3d). Exercise. (Apply Statement (3c) and (UO3).)

16. **Beyond simple arithmetic?**

We make two observations:

(a) Throughout the discussion above, we may replace every '*real*', '*real number*', '$\mathbb{R}$' et cetera respectively by '*rationals*', '*rational number*', '$\mathbb{Q}$' et cetera, and everything seems to remain fine.

(b) We seem to have touched upon nothing that may be related to infinitesimal calculus.

In fact (if we want to do so), we may formalize the 'rational number system' as understood in school maths in the exactly same way as we have done for the the real number system above. The **rational number system** will consist of:

- a set, denoted by $\mathbb{Q}$, whose elements are called **rational numbers**, amongst them two distinct rational numbers called **zero**, **one** and denoted by 0, 1 respectively,
- the arithmetic operations $+$, $\times$, called **addition, multiplication in the rationals** respectively) , and
- a subset of $\mathbb{Q}$, denoted by $\mathbb{Q}_{\geq 0}$, whose elements are called **non-negative rational numbers**,

forming an ordered field, in the sense that the **laws of arithmetic for the rationals** (which are the statements resultant from replacing '$\mathbb{R}$' in (A1)-(A11) by '$\mathbb{Q}$') and the **laws of order for the rationals (compatible to the arithmetic operations)** (which are the statements resultant from replacing '$\mathbb{R}$' in (O1)-(O3) by '$\mathbb{Q}$'), are true statements. An argument for a statement concerned with real numbers which relies on (A1)-(A11) and (O1)-(O3) alone will remain valid when adapted as an argument for the statement resultant from replacing '*real*' by '*rational*'.

This suggests why we have touched upon nothing that may be related to infinitesimal calculus. One crucial distinction between the rational number system and the real number system is that there are 'gaps' in the former while there is none in the latter. Infinitesimal calculus can be done in the real number system because of the non-existence of 'gaps' in the real number system. (This is technically known as **analytic completeness** of the real number system.)

With the introduction of the respective notions of greatest/least element, upper/lower bound, and supremum/infimum, we are ready to this question. (Refer to the handout *Greatest/least element, upper/lower bound.*) In terms of these notions, we can formulate the statement, assumed to be valid in the real number system, known as the **Least-upper-bound Axiom for the reals (LUBA)**:

(LUBA) *Let A be a non-empty subset of $\mathbb{R}$. Suppose A is bounded above in $\mathbb{R}$. Then A has a supremum in $\mathbb{R}$.*

It is with the help of the statement (LUBA) that we can construct the theory of calculus of one real variable. In your *mathematical analysis* course, the Least-upper-bound Axiom serves as the ultimate justification for other 'intuitively obvious' results which you have been using without questioning in *infinitesimal calculus*, such as the **Bounded-Monotone Theorem for infinite sequences of real numbers**, the **Intermediate Value Theorem** and the **Mean-Value Theorem**.

17. Although it also make sense to talk about respective notions of greatest/least element, upper/lower bound, and supremum/infimum in rational numbers (because they are formulated in terms of '$\leq$' alone), the statement below is not true:

    Let A be a non-empty subset of $\mathbb{Q}$. Suppose A is bounded above in $\mathbb{Q}$. Then A has a supremum in $\mathbb{Q}$.

    This signifies the key difference between the real number system and the rational number system. Calculus cannot be done with rational numbers alone.

18. Now we are ready to answer '*What is the real number system?*' fully.

    The **real number system** is made up of:—

    - a set, denoted by $\mathbb{R}$, whose elements are called **real numbers**, amongst them two distinct real numbers called **zero**, **one** and denoted by 0, 1 respectively,
    - the arithmetic operations $+$, $\times$, called **addition, multiplication in the reals** respectively) , and
    - a subset of $\mathbb{R}$, denoted by $\mathbb{R}_{\geq 0}$, whose elements are called **non-negative real numbers**,

    which form a **complete ordered field**, in the sense that the **Laws of Arithmetic for the reals**, namely the statements (A1)-(A11) and the **Laws of Order for the reals (compatible to the arithmetic operations)**, namely the statements (O1)-(O3), are true statements, and that the statement **Least-upper-bound Axiom**, namely the statement (LUBA), holds.

    This completes the formalization of the real number system as understood in school maths. The existence of such a set (the set of all real numbers) and the validity of the statements (A1)-(A11), (O1)-(O3) and (LUBA) (referred to as axioms of the real numer system) is the starting point in a standard *mathematical anaylsis* course.

19. Richard Dedekind, who was one of the nineteenth-century mathematicians to seriously study the nature of the real number system, chose the statement below as a 'fundamental assumption', in place of the Least-upper-bound Axiom.

Dedekind's Completeness Axiom.

Let A, B be subsets of $\mathbb{R}$. Suppose $A \cup B = \mathbb{R}$ and $A \cap B = \emptyset$. Further suppose that for any $\alpha \in A$, for any $\beta \in B$, $\beta < \alpha$. Then there exists some unique $\gamma \in \mathbb{R}$ such that $A \backslash \{\gamma\} = (\gamma, +\infty)$ and $B \backslash \{\gamma\} = (-\infty, \gamma)$.

Dedekind's Completeness Axiom turns out to be logically equivalent to the Least-upper-bound Axiom.

Weird though the statement looks, the idea in Dedekind's Completeness Axiom actually connects to the ancient Greeks' notion of ratios and proportions for arbitrary magnitudes, as described in Book V of Euclid's *Elements*. Dedekind described his investigation in '*Stetigkeit und irrationale Zahlen*' ('Continuity and irrational Numbers') and the preface of '*Was sind und was sollen die Zahlen?*' (Nature and meaning of numbers).

There are other statements which are logically equivalent to the Least-upper-bound Axiom. Many of them involve various notions of limiting behaviour of infinite sequences.

Definition. (Convergence of infinite sequence.)

Let $\{x_n\}_{n=0}^{\infty}$ be an infinite sequence. $\{x_n\}_{n=0}^{\infty}$ is said to be convergent in $\mathbb{R}$ if (there exists some $\ell \in \mathbb{R}$ such that for any $\varepsilon > 0$, (there exists some $N \in \mathbb{N}$ such that for any $n \in \mathbb{N}$, if $n > N$ then $|x_n - \ell| < \varepsilon$)).

The **Bounded-Monotone 'Theorem'** is one such statement which is logically equivalent to the Least-upper-bound Axiom.

Bounded-Monotone 'Theorem'.

Let $\{x_n\}_{n=0}^{\infty}$ be an infinite sequence of real numbers. Suppose $\{x_n\}_{n=0}^{\infty}$ is increasing. Further suppose $\{x_n\}_{n=0}^{\infty}$ is bounded above in $\mathbb{R}$. Then $\{x_n\}_{n=0}^{\infty}$ is convergent in $\mathbb{R}$.

There are others, such as the **Bolzano-Weierstrass 'Theorem'**.

20. From this point of view, the 'natural number system', the 'integer system', and the 'rational number system' are various 'subsystems' of the real number system, with underlying sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, being some special subsets of $\mathbb{R}$, described below, and with the arithmetic operations and the usual ordering in the real number syste restricted to these sets:

   (a) $\mathbb{N} = \left\{ x \in \mathbb{R} : x = \sum_{k=1}^{m} 1 \text{ for some } m \in \mathbb{N} \right\}$.

   (b) $\mathbb{Z} = \{x \in \mathbb{R} : x \in \mathbb{N} \text{ or } -x \in \mathbb{N}\}$.

   (c) $\mathbb{Q} = \{x \in \mathbb{R} : \text{There exist some } u, v \in \mathbb{Z} \text{ such that } v \neq 0 \text{ and u = xv}\}$

(In the symbol '$\sum_{k=1}^{m} 1$', the '1, m' above and below the summation symbol refer to the natural numbers 1, m respectively, while the the summand '1' refers to the real number 1. We understand $\sum_{k=1}^{0} 1$ as the real number 0. We then deliberately confuse each natural number m with the real number $\sum_{k=1}^{m} 1$.)