MATH1050 Answers to Examples: Cardinality.

1. ——

2. (a) Yes.

   (b) Yes.

3. (a) Yes.

   (b) Yes.

4. (a) An injective function from A_1 to B_1 is $f : A_1 \longrightarrow B_1$ defined by $f(x) = \dfrac{x}{2} + \dfrac{17}{6}$ for any $x \in A_1$.

   An injective function from B_1 to A_1 is $g : B_1 \longrightarrow A_1$ defined by $g(x) = x - 2$ for any $x \in B_1$.

   (b) An injective function from A_2 to B_2 is $f : A_2 \longrightarrow B_2$ by $f(x) = \dfrac{1}{x+2}$ for any $x \in A_2$.

   An injective function from B_2 to A_2 is $g : B_2 \longrightarrow A_2$ by $g(x) = x + 1$ for any $x \in B_2$.

   (c) An injective function from A_3 to B_3 is $f : A_3 \longrightarrow B_3$ defined by

$$f(x) = \begin{cases} e^x + 0.1 & \text{if} \quad x \in (-\infty, -1) \\ 1.1 + 0.5^x & \text{if} \quad x \in \mathbb{N} \end{cases}$$

   An injective function from B_3 to A_3 is $g : B_3 \longrightarrow A_3$ defined by $g(x) = x - 3$ for any $x \in B_3$.

   (d) An injective function from A_4 to B_4 is $f : A_4 \longrightarrow B_4$ defined by $f(x) = 0.0001x + 0.1$ for any $x \in A_4$.

   An injective function from B_4 to A_4 is $g : B_4 \longrightarrow A_4$ defined by $g(x) = 2^{-x} + 1$ for any $x \in B_4$.

   (e) An injective function from A_5 to B_5 is $f : A_5 \longrightarrow B_5$ by $f(x) = x + \sqrt{2}$ for any $x \in A_5$.

   An injective function from B_5 to A_5 is $g : B_5 \longrightarrow A_5$ by $g(y) = \dfrac{1}{y} + 1$ for any $y \in B_5$.

   (f) An injective function from D to S is the inclusion function $\iota : D \longrightarrow S$ given by $\iota(p) = p$ for any $p \in D$.

   An injective function from S to D is $g : S \longrightarrow D$ given by $g(z) = 0.5z$ for any $z \in S$.

5. (a) ——

   (b) ——

   Hint. One possible appropriate is to 'construct' the 'chain' $\mathbb{S}^2 \lesssim \mathbb{B}^3 \lesssim [-1,1]^3 \lesssim [-1,1] \lesssim \mathbb{S}^2$.

   (c) ——

   Hint. One possible approach is to construct the 'chain' $\mathbb{S}^1 \lesssim \mathbb{S}^2 \lesssim [-1,1]^3 \lesssim [-1,1] \lesssim \mathbb{S}^1$.

6. ——

7. ——

8. (a) An injective function from $A \cup B$ to $(A \times, \{0\}) \cup (B \times \{1\})$ is given by the relation $\psi = (A \cup B, (A \times \{0\}) \cup (B \times \{1\}), H)$defined by

$$H = \{(x, (x, 0)) \mid x \in A\} \cup \{(y, (y, 1)) \mid y \in B \backslash A\}.$$

   (b) A bijective function from $A \times \{0\}$ to $A \times \{b\}$ is given by the relation $\varphi_1 = (A \times \{0\}, A \times \{b\}, G_1)$ defined by by $G_1 = \{((x, 0), (x, b)) \mid x \in A\}$.

   (c) A bijective function from $B \times \{1\}$ to $(\{a\} \times (B \backslash \{b\})) \cup \{(a', b')\}$ is given by the relation $\varphi_2 = (B \times \{1\}, (\{a\} \times (B \backslash \{b\})) \cup \{(a', b')\}, G_2)$ defined by

$$G_2 = \{((y, 1), (a, y)) \mid y \in B \backslash \{b\}\} \cup \{((b, 1), (a', b'))\}$$

   (d) The relation

$$\varphi = ((A \times \{0\}) \cup (B \times \{1\}), (A \times \{b\}) \cup (\{a\} \times (B \backslash \{b\})) \cup \{(a', b')\}, G_1 \cup G_2)$$

   is a bijective function.

   (e) ——

9. (a) ——
   (b) ——
   (c) Yes.
10. (a) ——
    (b) ——
    (c) Yes.
11. ——
12. (a) ——
    (b) *Hint.* The function given by $\Psi : \mathsf{Map}(\mathbb{N}, \{0,1,2\}) \longrightarrow \mathsf{Map}(\mathbb{N}, \{0,1\})$ by

$$(\Psi(\varphi))(n) = \begin{cases} 1 & \text{if} \quad n = 3k \text{ and } \varphi(k) = 0 \\ 0 & \text{if} \quad n = 3k \text{ and } \varphi(k) = 1 \\ 0 & \text{if} \quad n = 3k \text{ and } \varphi(k) = 2 \\ 0 & \text{if} \quad n = 3k+1 \text{ and } \varphi(k) = 0 \\ 1 & \text{if} \quad n = 3k+1 \text{ and } \varphi(k) = 1 \\ 0 & \text{if} \quad n = 3k+1 \text{ and } \varphi(k) = 2 \\ 0 & \text{if} \quad n = 3k+2 \text{ and } \varphi(k) = 0 \\ 0 & \text{if} \quad n = 3k+2 \text{ and } \varphi(k) = 1 \\ 1 & \text{if} \quad n = 3k+2 \text{ and } \varphi(k) = 2 \end{cases}$$

    is an injective function from $\mathsf{Map}(\mathbb{N}, \{0,1,2\})$ to $\mathsf{Map}(\mathbb{N}, \{0,1\})$
13. ——
14. ——
15. ——
16. ——

    Remark. In fact, we have $C(J) \sim C^n(J)$, where $C^n(J)$ the set of all real-valued n-th times differentiable functions on J whose n-th derivatives are continuous functions on J for each $n \in \mathbb{N} \backslash \{0\}$.
17. (a) $\mathfrak{F}_0(\mathbb{N}) = \{\emptyset\}$.
    (b) The function $f_1 : \mathbb{N} \longrightarrow \mathfrak{F}_1(\mathbb{N})$ given by $f_1(x) = \{x\}$ for any $x \in \mathbb{N}$ is a bijective function from $\mathbb{N}$ to $\mathfrak{F}_1(\mathbb{N})$.
    (c) The function $f_2 : \mathbb{N}^2 \longrightarrow \mathfrak{F}_2(\mathbb{N}) \cup \mathfrak{F}_1(\mathbb{N})$ given by $f_2(x, y) = \{x, y\}$ for any $x, y \in \mathbb{N}$ is a surjective function from $\mathbb{N}^2$ to $\mathfrak{F}_2(\mathbb{N}) \cup \mathfrak{F}_1(\mathbb{N})$.
    (d) Yes.
    (e) Yes.
    (f) Yes.
    (g) Yes.
    (h) No.
18. (a) Yes. The cardinality of $\mathfrak{S}_n(A)$ is $|A|^n$.
    (b) Yes.
    (c) No.