MATH1050 Examples: Cardinality.

1. (a) Verify that $2^x(2y+1) \in \mathbb{N}\backslash\{0\}$ for any $x, y \in \mathbb{N}$.
   (b) Define the function $f : \mathbb{N}^2 \longrightarrow \mathbb{N}\backslash\{0\}$ by $f(x, y) = 2^x(2y+1)$ for any $x, y \in \mathbb{N}$.
   Verify that f is bijective.
   (c) Verify that $\mathbb{N}^2 \sim \mathbb{N}$.

2. Let $S = \{x \in \mathbb{N} : x = m^2 \text{ for some } m \in \mathbb{N}\}$, $C = \{y \in \mathbb{N} : y = n^3 \text{ for some } n \in \mathbb{N}\}$.

   Define $F = \left\{ (x, y) \;\middle|\; \begin{array}{l} x \in S \text{ and } y \in C \text{ and} \\ \text{there exists some } k \in \mathbb{N} \\ \text{such that } (x = k^2 \text{ and } y = k^3). \end{array} \right\}$, and $f = (S, C, F)$. Note that $F \subset S \times C$.

   (a) Is f a function from S to C? Justify your answer.
   (b) Is it true that $S \sim C$? Justify your answer.

3. Let p, q be distinct positive odd integers, and

$$A = \{x \in \mathbb{Q} : x = s^p \text{ for some } s \in \mathbb{Q}\}, \quad B = \{y \in \mathbb{Q} : y = t^q \text{ for some } t \in \mathbb{Q}\}$$

   Define $F = \left\{ (x, y) \;\middle|\; \begin{array}{l} x \in A \text{ and } y \in B \text{ and} \\ \text{there exists some } r \in \mathbb{Q} \\ \text{such that } (x = r^p \text{ and } y = r^q). \end{array} \right\}$ and $f = (A, B, F)$. Note that $F \subset A \times B$.

   (a) Is f a function from A to B? Justify your answer.
   (b) Is it true that A is of cardinality equal to B? Justify your answer.

4. (a) Let $A_1 = [1, 2]$, $B_1 = (3, 4)$. Apply the Schröder-Bernstein Theorem to prove that $A_1 \sim B_1$.
   (b) Let $A_2 = [0, +\infty)$, $B_2 = (-1, 1) \cup [2, 3]$. Apply the Schröder-Bernstein Theorem to prove that $A_2 \sim B_2$.
   (c) Let $A_3 = (-\infty, -1) \cup \mathbb{N}$, $B_3 = [0.1, 0.9] \cup (1.1, 1.9)$. Apply the Schröder-Bernstein Theorem to prove that $A_3 \sim B_3$.
   (d) Let $A_4 = [1, 9] \cup (\mathbb{Q} \cap [10, 99])$, $B_4 = (0.01, 0.09) \cup (0.1, 0.9) \cup \mathbb{N}$. Apply the Schröder-Bernstein Theorem to prove that $A_4 \sim B_4$.
   (e) Let $A_5 = [1, 2] \cup \{100\}$ and $B_5 = (1, 10) \cup ((100, +\infty)\backslash\mathbb{Q})$. Apply the Schröder-Bernstein Theorem to prove that $A_5 \sim B_5$.
   (f) Let $D = \{\zeta \in \mathbb{C} \mid |\zeta| \leq 1\}$, $S = \{\zeta \in \mathbb{C} : |\mathsf{Re}(\zeta)| \leq 1 \text{ and } |\mathsf{Im}(\zeta)| \leq 1\}$. Apply the Schröder-Bernstein Theorem to prove that $D \sim S$.

5. *In this question, you may take for granted the results* $[0, 1] \sim \mathbb{R}$, $[0, 1] \sim [0, 1]^2$, $\mathbb{R} \sim \mathbb{R}^2$.

   (a) Let Π be the set of all planes in $\mathbb{R}^3$. Apply the Schröder-Bernstein Theorem to prove that $\Pi \sim \mathbb{R}$.
   Remark. Let Λ be the set of all lines in $\mathbb{R}^3$. How to prove $\Lambda \sim \mathbb{R}$?
   (b) Let $\mathbb{S}^2 = \{(x, y, z) \mid x, y, z \in \mathbb{R} \text{ and } x^2 + y^2 + z^2 = 1\}$, $\mathbb{B}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R} \text{ and } x^2 + y^2 + z^2 \leq 1\}$. Apply the Schröder-Bernstein Theorem to prove that $\mathbb{S}^2 \sim \mathbb{B}^3$.
   (c) Let $\mathbb{S}^1 = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$, $\mathbb{S}^2 = \{(x, y, z) \mid x, y, z \in \mathbb{R} \text{ and } x^2 + y^2 + z^2 = 1\}$. Apply the Schröder-Bernstein Theorem to prove that $\mathbb{S}^1 \sim \mathbb{S}^2$.

6. Let A, B, C be sets. Prove the statements below:

   (a) $A \sim A$.
   (b) *Suppose* $A \sim B$. *Then* $B \sim A$.
   (c) *Suppose* $A \sim B$ *and* $B \sim C$. *Then* $A \sim C$.
   (d) $A \lesssim A$.
   (e) *Suppose* $A \lesssim B$ *and* $B \lesssim C$. *Then* $A \lesssim C$.

7. (a) Let A, B, C, D be sets, and $f : A \longrightarrow C$, $g : B \longrightarrow D$ be functions. Define the function $f \times g : A \times B \longrightarrow C \times D$ by $(f \times g)(x, y) = (f(x), g(y))$ for any $x \in A$, for any $y \in B$.

i. Suppose f, g are surjective. Verify that $f \times g$ is surjective.
ii. Suppose f, g are injective. Verify that $f \times g$ is injective.
iii. Suppose f, g are bijective. Verify that $f \times g$ is bijective.

(b) Let A, B, C, D be sets. Suppose $A \sim C$ and $B \sim D$. Prove that $A \times B \sim C \times D$.

Remark. Hence the statement below holds:

Let A, B be sets. Suppose $A \sim B$. Then $A^2 \sim B^2$.

8. Let A, B be non-empty sets. Suppose each of A, B is not a singleton. Pick $a, a' \in A$, with $a \neq a'$, and pick $b, b' \in B$, with $b \neq b'$. Regard $0, 1$ as distinct objects.

(a) Construct an injective function from $A \cup B$ to $(A \times \{0\}) \cup (B \times \{1\})$.

(b) Construct a bijective function from $A \times \{0\}$ to $A \times \{b\}$.

(c) Construct a bijective function from $B \times \{1\}$ to $(\{a\} \times (B \backslash \{b\})) \cup \{(a', b')\}$.

(d) Construct a bijective function from $(A \times \{0\}) \cup (B \times \{1\})$ to $(A \times \{b\}) \cup (\{a\} \times (B \backslash \{b\})) \cup \{(a', b')\}$.

(e) Conclude that $A \cup B \lesssim A \times B$.

9. Let A be a non-empty set. Define the function $\chi : \mathfrak{P}(A) \longrightarrow \mathsf{Map}(A, \{0, 1\})$ by $\chi(S) = \chi_S^A$ for any $S \in \mathfrak{P}(A)$.

Here, for any subset S of A, $\chi_S^A : A \longrightarrow \{0, 1\}$ is the characteristic function of S in A, defined by

$$\chi_S^A(x) = \begin{cases} 1 & \text{if} \quad x \in S, \\ 0 & \text{if} \quad x \in A \backslash S. \end{cases}$$

(a) Verify that χ is surjective.

(b) Verify that χ is injective.

(c) Is it true that $\mathfrak{P}(A) \sim \mathsf{Map}(A, \{0, 1\})$? Justify your answer.

10. Define the functions $\sigma, \tau : \mathbb{N} \longrightarrow \mathbb{N}$ by $\sigma(n) = 2n$, $\tau(n) = 2n + 1$ for any $n \in \mathbb{N}$.

Let B be a non-empty set. Define the function $f : \mathsf{Map}(\mathbb{N}, B) \longrightarrow (\mathsf{Map}(\mathbb{N}, B))^2$ by $f(\varphi) = (\varphi \circ \sigma, \varphi \circ \tau)$ for any $\varphi \in \mathsf{Map}(\mathbb{N}, B)$.

(a) Verify that f is surjective.

(b) Verify that f is injective.

(c) Is it true that $\mathsf{Map}(\mathbb{N}, B) \sim (\mathsf{Map}(\mathbb{N}, B))^2$? Justify your answer.

11. *In this question, we are going to give a proof for the Schröder-Bernstein Theorem.*

(a) Let A, B be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow A$ be injective functions.
For any subset V of B, define $V^* = B \backslash f(A \backslash g(V))$. (Note that V^* is a subset of B.)
Define $\mathfrak{C} = \{V \in \mathfrak{P}(B) : V^* \subset V\}$, $K = \{y \in B : y \in V \text{ for any } V \in \mathfrak{C}\}$.
Prove the statements below:

i. *For any subsets V, W of B, if $V \subset W$ then $V^* \subset W^*$.*

ii. $K \in \mathfrak{C}$.

Remark. This is a hint: *By the definition of K, we have $K \subset W$ for any $W \in \mathfrak{C}$.*

iii. $K^* = K$.

iv. $f(A \backslash g(K)) = B \backslash K$.

(b) Apply the above results to prove the Schröder-Bernstein Theorem.

Remark. How to start the argument? Focus on what part (a.iv) suggests for a pair of injective functions whose respective domains are the respective ranges of the others. At some stage of the subsequent argument, you may need the Glueing Lemma.

12. (a) Define the function $\Phi : \mathsf{Map}(\mathbb{N}, \{0, 1\}) \longrightarrow \mathsf{Map}(\mathbb{N}, \{0, 1, 2\})$ by $(\Phi(\alpha))(x) = \alpha(x)$ for any $x \in \mathbb{N}$.
Verify that Φ is an injective function.

(b) Apply the Schröder-Bernstein Theorem, or otherwise, to prove that $\mathsf{Map}(\mathbb{N}, \{0, 1\}) \sim \mathsf{Map}(\mathbb{N}, \{0, 1, 2\})$.

13. (a) Let A, B, C, D be non-empty sets. Prove the statements below:

i. *Suppose* $A \sim C$ *and* $B \sim D$. *Then* $\mathsf{Map}(A,B) \sim \mathsf{Map}(C,D)$.

ii. *Suppose* $A \subset C$. *Then* $\mathsf{Map}(A,B) \lesssim \mathsf{Map}(C,B)$.

iii. *Suppose* $B \subset D$. *Then* $\mathsf{Map}(A,B) \lesssim \mathsf{Map}(A,D)$.

iv. *Suppose* $B \lesssim D$. *Then* $\mathsf{Map}(A,B) \lesssim \mathsf{Map}(A,D)$.

v. *Suppose* $A \subset C$ *and* $B \subset D$. *Then* $\mathsf{Map}(A,B) \lesssim \mathsf{Map}(C,D)$.

vi. $\mathsf{Map}(A \times B, C) \sim \mathsf{Map}(A, \mathsf{Map}(B,C))$.

(b) Prove each of the statements below. Where necessary, apply the Schröder-Bernstein Theorem. You may take for granted that $\mathbb{N}^2 \sim \mathbb{N}$, $\mathbb{R}^2 \sim \mathbb{R}$ and $\mathbb{R} \sim \mathsf{Map}(\mathbb{N}, [\![0,9]\!])$.

i. $\mathsf{Map}(\mathbb{N}, \{0,1\}) \lesssim \mathsf{Map}(\mathbb{N}, \mathbb{N})$.

ii. $\mathsf{Map}(\mathbb{N}, \mathbb{N}) \lesssim \mathsf{Map}(\mathbb{N}, \mathsf{Map}(\mathbb{N}, \{0,1\}))$.

iii. $\mathsf{Map}(\mathbb{N}, \mathbb{N}) \sim \mathsf{Map}(\mathbb{N}, \{0,1\})$.

iv. $\mathbb{R} \sim \mathsf{Map}(\mathbb{N}, \mathbb{N})$.

v. $\mathsf{Map}(\mathbb{R}, \{0,1\}) \sim \mathsf{Map}(\mathbb{R}, \mathbb{N})$.

vi. $\mathsf{Map}(\mathbb{R}, \mathbb{N}) \sim \mathsf{Map}(\mathbb{R}, \mathbb{R})$.

14. *In this question, we are going to give another proof for Cantor's Theorem on the power set of any given set.*

(a) Let A be a set, and $f : A \longrightarrow \mathfrak{P}(A)$ be a function. Define $C_f = \{x \in A : x \notin f(x)\}$. (Note that $C_f \in \mathfrak{P}(A)$.)

i. Dis-prove the statement '*there exists some* $z \in A$ *such that* $f(z) = C_f$.

ii. Hence deduce that f is not surjective.

Remark. The set C_f is called **Cantor's diagonal set for the function f.**

(b) Apply the above results to prove Cantor's Theorem on the power set of any given set.

15. We introduce/recall the definitions below:

- *Let* $z \in \mathbb{C}$.
  - z *is said to be a* **Gaussian rational number** *if both of* $\mathsf{Re}(z), \mathsf{Im}(z)$ *are rational numbers.*
  - z *is said to be a* **Gaussian irrational number** *if* z *is not a Gaussian rational number.*

The set of all Gaussian rational numbers is denoted by $\mathbb{Q}[i]$.

For any $p, q \in \mathbb{C}$, we define $\sigma[p,q]$ to be the set $\{\tau p + (1-\tau)q \mid \tau \in [0,1]\}$. ($\sigma[p,q]$ is the line segment on the Argand plane joining the point p and the point q.)

Let $z_1, z_2 \in \mathbb{C} \backslash \mathbb{Q}[i]$. Suppose $z_1 \neq z_2$. Prove that there exist some $w \in \mathbb{C} \backslash \mathbb{Q}[i]$ such that the $\sigma[z_1, w] \cup \sigma[z_2, w] \subset \mathbb{C} \backslash \mathbb{Q}[i]$.

Remark. Hence any two Gaussian irrational numbers can be joint by a path made up of two line segments which lie entirely in the set of Gaussian irrational numbers. The proof-by-contradiction method is more suitable for the argument for this result. At some stage of the argument you may need the result $\mathbb{N} < \mathbb{R}$ (or something equivalent) and the Schröder-Bernstein Theorem.

16. *Familiarity with the calculus of one variable is assumed in this question.*

Let J be an open interval in $\mathbb{R}$. Denote by $C(J)$ the set of all real-valued continuous functions on J. Denote by $C^1(J)$ the set of all real-valued differentiable functions on J whose first derivatives are continuous functions on J.

Apply the Schröder-Bernstein Theorem, or otherwise, to prove that $C(J) \sim C^1(J)$.

17. Consider the sets $\mathbb{N}$ and $\mathfrak{P}(\mathbb{N})$. We introduce these notations:

- *We write* $\mathfrak{F}(\mathbb{N}) = \{S \in \mathfrak{P}(\mathbb{N}) : S \text{ is finite.}\}$. ($\mathfrak{F}(\mathbb{N})$ is the set of all finite subsets of $\mathbb{N}$.)
- *For any* $n \in \mathbb{N}$, *we write* $\mathfrak{F}_n(\mathbb{N}) = \{S \in \mathfrak{P}(\mathbb{N}) : S \text{ is finite and } |S| = n.\}$. ($\mathfrak{F}_n(\mathbb{N})$ is the set of all subsets of cardinality n of $\mathbb{N}$. It is by definition a subset of $\mathfrak{F}(\mathbb{N})$.)
- *We write* $\mathfrak{C}_\infty(\mathbb{N}) = \{S \in \mathfrak{P}(\mathbb{N}) : S \text{ is countably infinite.}\}$. ($\mathfrak{C}_\infty(\mathbb{N})$ is the set of all countably infinite subsets of $\mathbb{N}$.)

Note that the statements below hold:

(A) $\mathfrak{F}(\mathbb{N}) \cup \mathfrak{C}_\infty(\mathbb{N}) = \mathfrak{P}(\mathbb{N})$.

(B) $\mathfrak{F}(\mathbb{N}) \cap \mathfrak{C}_\infty(\mathbb{N}) = \emptyset$.

(C) $\mathfrak{F}(\mathbb{N}) = \{S \in \mathfrak{F}(\mathbb{N}) : S \in \mathfrak{F}_n(\mathbb{N}) \text{ for some } n \in \mathbb{N}\}$.

(D) $\mathfrak{F}_m(\mathbb{N}) \cap \mathfrak{F}_n(\mathbb{N}) = \emptyset$ *whenever* $m \neq n$.

These combine together to give the formal formulation of the 'fact' that $\mathfrak{P}(\mathbb{N})$ is 'partitioned' into these 'infinitely many' 'chambers': the set of all (countably) infinite subsets of $\mathbb{N}$, the set of all (finite) subsets of $\mathbb{N}$ with one element, the set of all (finite) subsets of $\mathbb{N}$ with two elements, the set of all (finite) subsets of $\mathbb{N}$ with three elements, ...

(a) What is $\mathfrak{F}_0(\mathbb{N})$?

(b) Write down a bijective function from $\mathbb{N}$ to $\mathfrak{F}_1(\mathbb{N})$.

(c) Write down a surjective function from $\mathbb{N}^2$ to $\mathfrak{F}_2(\mathbb{N}) \cup \mathfrak{F}_1(\mathbb{N})$.

(d) Is there an injective function from $\mathfrak{F}_2(\mathbb{N})$ to $\mathbb{N}^2$? Justify your answer.

(e) Is there an injective function from $\mathfrak{F}_3(\mathbb{N})$ to $\mathbb{N}^3$? Justify your answer.

(f) Is it true that $\mathfrak{F}_n(\mathbb{N})$ is countable for any $n \in \mathbb{N}$? Justify your answer.

(g) Is it true that $\mathfrak{F}(\mathbb{N})$ is countable? Justify your answer.

(h) Is $\mathfrak{C}_\infty(\mathbb{N})$ countable? Justify your answer.

18. Let A be a non-empty finite set. We introduce these notations:

- *We write* $\mathfrak{S}(A) = \bigcup_{n=0}^{\infty} \mathsf{Map}([\![1, n]\!], A)$. ($\mathfrak{S}(A)$ is the set of all finite sequences in A. Read $\bigcup_{n=0}^{\infty} \mathsf{Map}([\![1, n]\!], A)$ as $\{\varphi \mid \varphi \in \mathsf{Map}([\![1, n]\!], A) \text{ for some } n \in \mathbb{N}\}$.)
- *For any* $n \in \mathbb{N}$, *we write* $\mathfrak{S}_n(A) = \mathsf{Map}([\![1, n]\!], A)$. ($\mathfrak{S}_n(A)$ is the set of all finite sequences of length n in A.)

(a) Let $n \in \mathbb{N}$. Is $\mathfrak{S}_n(A)$ finite? If it is finite, what is its cardinality?

(b) Is $\mathfrak{S}(A)$ countably infinite? Why?

(c) Is there any surjective function from $\mathfrak{S}(A)$ to $\mathsf{Map}(\mathfrak{S}(A), \mathfrak{S}(A))$? Why?