1. **Definition.**
   Let A be a set.
   (1) *A is* **countable** *if $A \lesssim \mathbb{N}$.*
   (2) *A is said to be* **countably infinite** *if $A \sim \mathbb{N}$.*
   (3) *A is said to be* **uncountable** *if A is not countable.*

   Basic examples of countably infinite sets.

   (a) $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$;
   (b) $\mathbb{N}^2$, $\mathbb{N}^3$, $\mathbb{N}^4$, ...

   Basic examples of uncountable sets.

   (a) $\mathsf{Map}(\mathbb{N}, \{0,1\})$, $\mathsf{Map}(\mathbb{N}, [\![0,9]\!])$, $\mathsf{Map}(\mathbb{N}, \mathbb{N})$;
   (b) $[0,1]$, $\mathbb{R}$, $\mathbb{C}$;
   (c) $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^4$, ... ;
   (d) $\mathfrak{P}(\mathbb{N}), \mathfrak{P}(\mathfrak{P}(\mathbb{N}))$, $\mathfrak{P}(\mathfrak{P}(\mathfrak{P}(\mathbb{N})))$, ...

   Theorem (XXI).

   (1) *Suppose A is a set. Then A is countable iff (A is finite or A is countably infinite).*
   (2) *Suppose A is a set. Then A is countably infinite iff A is both countable and infinite.*
   (3) *A is uncountable iff $\mathbb{N} < A$.*

   Remark.
   The arguments of the respective statements are word games, possibly involving the application of Schröder-Bernstein Theorem.
   Further remarks.

   (a) Heuristic idea on ‘being countable’: A is countable exactly when we can identify A as a subset of $\mathbb{N}$ by labeling the elements of A exhaustively by natural numbers.
   (b) Heuristic idea on ‘being uncountable’: A is uncountable exactly when there are ‘so many’ elements in A that there is no way to label all the elements of A by natural numbers alone.
   (c) We may think of $\mathbb{N}$ or whatever countably infinite set as a ‘smallest’ infinite set.

   Classification of sets by comparing ‘relative sizes’ with the ‘smallest’ infinite set $\mathbb{N}$:

A is finite. ($A < \mathbb{N}$.)	A is infinite. ($\mathbb{N} \lesssim A$.)	*same as left.*
same as above.	A is countably infinite. ($A \sim \mathbb{N}$.)	*same as below.*
same as right.	A is countable. ($A \lesssim \mathbb{N}$.)	A is uncountable. ($\mathbb{N} < A$.)

2. **‘Hilbert’s Hotel’.**
   Every ‘infinite’ set can ‘absorb’ a countably infinite set (or a finite set, which is a set of ‘smaller size’ than $\mathbb{N}$) to form a new set of the ‘same size’ as itself.
   Theorem (XXII). (‘Hilbert’s Hotel’.)
   Let A, B be sets. Suppose A is infinite and B is countable. Further suppose $A \cap B = \emptyset$. Then $A \cup B \sim A$.
   Proof. The result follows from Theorem (XXI), Lemma (XXIII) and Lemma (XXIV).
   Lemma (XXIII).
   Let A, B be sets. Suppose A is infinite and B is finite. Further suppose $A \cap B = \emptyset$. Then $A \cup B \sim A$.
   Proof.
   Let A, B be sets. Suppose A is infinite and B is finite. Further suppose $A \cap B = \emptyset$.
   There is an injective function from $\mathbb{N}$ to A. This injective function defines some infinite sequence $\{x_n\}_{n=0}^{\infty}$ in A with no repeated terms: $x_m \neq x_n$ whenever $m \neq n$.
   Write $S = \{x_n \mid n \in \mathbb{N}\}$.
   Since B is finite, and $|B| = N$ for some $N \in \mathbb{N}$. Write $B = \{y_0, y_1, \cdots, y_{N-1}\}$.

Idea. How to proceed with the construction of a bijective function from A to $A \cup B$?
The table below gives the idea:

A	S	x_0	x_1	$\cdots$	x_{N-1}	x_N	x_{N+1}	x_{N+2}	$\cdots$	$z \in A \backslash S$
		$\downarrow$	$\downarrow$	$\cdots$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\cdots$	$\downarrow$
$A \cup B$	$S \cup B$	y_0	y_1	$\cdots$	y_{N-1}	x_0	x_{0+1}	x_{0+2}	$\cdots$	$z \in A \backslash S$

Define [with the table above in mind]

$$\begin{aligned} F_1 &= \{(x_i, y_i) \mid i \in [\![0, N-1]\!]\} \cup \{(x_j, x_{j-N}) \mid j \in \mathbb{N} \text{ and } j \geq N\}, \\ F_2 &= \{(z, z) \mid z \in A \backslash S\}. \end{aligned}$$

Note that $F_1 \subset A \times (A \cup B)$ and $F_2 \subset A \times (A \cup B)$.

Define the relation $f = (A, A \cup B, F)$ by $F = F_1 \cup F_2$.

f is a bijective function from A to $A \cup B$. (Why?) It follows that $A \sim A \cup B$.

Lemma (XXIV).

Let A, B be sets. Suppose A is infinite and B is countably infinite. Further suppose $A \cap B = \emptyset$. Then $A \cup B \sim A$.

Proof.

Let A, B be sets. Suppose A is infinite and B is countably infinite. Further suppose $A \cap B = \emptyset$.

There is an injective function from $\mathbb{N}$ to A. This injective function defines some infinite sequence $\{x_n\}_{n=0}^{\infty}$ in A with no repeated terms: $x_m \neq x_n$ whenever $m \neq n$.

Write $S = \{x_n \mid n \in \mathbb{N}\}$.

Since $B \sim \mathbb{N}$, there is a bijective function from $\mathbb{N}$ to B. This bijective function defines some infinite sequence $\{y_n\}_{n=0}^{\infty}$ in B, exhausting B and with no repeated terms: $B = \{y_n \mid n \in \mathbb{N}\}$, and $y_m \neq y_n$ whenever $m \neq n$.

Idea. How to proceed with the construction of a bijective function from A to $A \cup B$? This is the idea:

A	S	x_0	x_1	x_2	x_3	x_4	x_5	$\cdots$	x_{2N}	x_{2N+1}	$\cdots$	$z \in A \backslash S$
		$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\cdots$	$\downarrow$	$\downarrow$	$\cdots$	$\downarrow$
$A \cup B$	$S \cup B$	y_0	x_0	y_1	x_1	y_2	x_2	$\cdots$	y_N	x_N	$\cdots$	$z \in A \backslash S$

Define [with the table above in mind]

$$\begin{aligned} G_1 &= \{(x_{2n}, y_n) \mid n \in \mathbb{N}\} \cup \{(x_{2n+1}, x_n) \mid n \in \mathbb{N}\}, \\ G_2 &= \{(z, z) \mid z \in A \backslash S\} \end{aligned}$$

Note that $G_1 \subset A \times (A \cup B)$ and $G_2 \subset A \times (A \cup B)$.

Define the relation $g = (A, A \cup B, G)$ by $G = G_1 \cup G_2$.

g is a bijective function from A to $A \cup B$. (Why?) It follows that $A \sim A \cup B$.

3. **Consequences of 'Hilbert's Hotel'.**

The conclusion in Theorem (XXII) still holds when the condition '$A \cap B = \emptyset$' is dropped.

Corollary (XXV).

Let C, D be sets. Suppose C is infinite and D is countable. Then $C \cup D \sim C$.

Proof. Exercise. (Apply Theorem (XXII).)

Corollary (XXVI).

Let C, D be sets. Suppose $D \subset C$. Also suppose that $C \backslash D$ is infinite and D is countable. Then $C \backslash D \sim C$.

Proof. Exercise. (Apply Theorem (XXII).)

Example of application of Corollary (XXVI).

$\mathbb{R} \backslash \mathbb{Q} \sim \mathbb{R}$. Also, $\mathbb{Q} < \mathbb{R} \backslash \mathbb{Q}$. Justification:

$\{\sqrt[n+2]{2}\}_{n=0}^{\infty}$ is an infinite sequence with no repeated terms in $\mathbb{R} \backslash \mathbb{Q}$. (Why?) Therefore $\mathbb{R} \backslash \mathbb{Q}$ is infinite. By Corollary (XXVI), $\mathbb{R} \backslash \mathbb{Q} \sim \mathbb{R}$. Since $\mathbb{Q} < \mathbb{R}$, we also have $\mathbb{Q} < \mathbb{R} \backslash \mathbb{Q}$.

Remark. There are as many irrational numbers as real numbers. There are 'much more' irrational numbers than rational numbers.

4. **Countable union of countable sets.**

Recall the definition for the notion of generalized union:

- *Let M be a set and $\{S_n\}_{n=0}^{\infty}$ be an infinite sequence of subsets of M. The (generalized) union of $\{S_n\}_{n=0}^{\infty}$ is defined to be the set $\{x \in M : x \in S_n$ for some $n \in \mathbb{N}\}$. It is denoted by $\bigcup_{n=0}^{\infty} S_n$.*

Theorem (XXVII). (Countability of countable union of countable sets.)

Let A be a set. Suppose $\{A_n\}_{n=0}^{\infty}$ is an infinite sequence of countable subsets of A. Then $\bigcup_{n=0}^{\infty} A_n$ is countable.

Remark. Hence 'the (generalized) union of countably many countable sets is countable'.

Proof.

- *Idea of the argument.*

  Write $B = \bigcup_{n=0}^{\infty} A_n$.

  For each $n \in \mathbb{N}$, label the elements of A_n exhaustively with elements of $\mathbb{N}$, so that we have $A_n = \{x_{n0}, x_{n1}, x_{n2}, \cdots\}$.
  Now obtain the possibly 'infinite array', which exhausts the elements of the set B:

$$\begin{array}{c|ccccc} A_0 & x_{00} & x_{01} & x_{02} & x_{03} & \cdots \\ A_1 & x_{10} & x_{11} & x_{12} & x_{13} & \cdots \\ A_2 & x_{20} & x_{21} & x_{22} & x_{23} & \cdots \\ A_3 & x_{30} & x_{31} & x_{32} & x_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}$$

  For each n, the 'row' of the 'x_{nj}'s may 'terminate' or not. There may be 'repeated' entries amongst the 'x_{ij}'s'. No matter what, the 'size' of this 'array' is at most that of $\mathbb{N}^2$ and hence that of $\mathbb{N}$. Hence B is countable.

- *Formal argument.* Exercise.

Corollary (XXVIII). (Sufficiency criteria for being countably infinite.)

Let A be a set, and $\{A_n\}_{n=0}^{\infty}$ be an infinite sequence of countable subsets of A.

(1) *Suppose $\bigcup_{n=0}^{\infty} A_n$ is infinite. Then $\bigcup_{n=0}^{\infty} A_n$ is countably infinite.*

(2) *Suppose there exists some $m \in \mathbb{N}$ such that A_m is countably infinite. Then $\bigcup_{n=0}^{\infty} A_n$ is countably infinite.*

(3) *Suppose there exists some infinite sequence $\{x_n\}_{n=0}^{\infty}$ in A such that both of the statements below hold:*

  (3a) *($x_n \in A_n$ for any $n \in \mathbb{N}$) and*

  (3b) *(for any $k, m \in \mathbb{N}$, if $k \neq m$ then $x_k \neq x_m$).*

  Then $\bigcup_{n=0}^{\infty} A_n$ is countably infinite.

5. **Examples of applications of Theorem (XXVII), Theorem (XXVIII)**.

(1) Another argument for $\mathbb{Q} \sim \mathbb{N}$:

- Write $\mathbb{N}^* = \mathbb{N} \backslash \{0\}$.
  For any $n \in \mathbb{N}^*$, define $Q_n = \left\{ \frac{m}{n} \middle| m \in \mathbb{Z} \right\}$. We have $\mathbb{Q} = \bigcup_{n=1}^{\infty} Q_n$.
  Note that $Q_n \sim \mathbb{Z} \sim \mathbb{N}$. Then $\mathbb{Q} \lesssim \mathbb{N}$.
  Recall $\mathbb{N} \lesssim \mathbb{Q}$. Then we have $\mathbb{Q} \sim \mathbb{N}$.

(2) Denote by $\mathbb{Q}[x]$ the set of all polynomials with indeterminate x and with coefficients in $\mathbb{Q}$.
$\mathbb{Q}[x] \backslash \{0\} \sim \mathbb{N}$. Also, $\mathbb{Q}[x] \sim \mathbb{N}$. Justification:

- For any $n \in \mathbb{N}$, define $T_n = \{f(x) \in \mathbb{Q}[x] : \deg(f(x)) = n\}$.
  We have $\mathbb{Q}[x] \backslash \{0\} = \bigcup_{n=0}^{\infty} T_n$.
  Write $\mathbb{Q}^* = \mathbb{Q} \backslash \{0\}$. Note that $T_n \sim \mathbb{Q}^* \times \mathbb{Q}^n \sim \mathbb{N}$. Then $\mathbb{Q}[x] \backslash \{0\} \lesssim \mathbb{N}$.
  For each $j \in \mathbb{N}$, denote by x^j the monic monomial of degree j.
  $\mathbb{N} \sim \{x^j \mid j \in \mathbb{N}\} \lesssim \mathbb{Q}[x] \backslash \{0\}$. Then we have $\mathbb{Q}[x] \backslash \{0\} \sim \mathbb{N}$.
  Then $\mathbb{Q}[x] \sim \mathbb{N}$ also. (Why?)

(3) Define $\mathbb{A} = \{\zeta \in \mathbb{C} : \zeta \text{ is a root of } f(x) \text{ for some } f(x) \in \mathbb{Q}[x] \backslash \{0\}\}$.
The elements of $\mathbb{A}$ are called **algebraic numbers (over $\mathbb{Q}$)**. The elements of $\mathbb{C} \backslash \mathbb{A}$ are called **transcendental numbers (over $\mathbb{Q}$)**.
$\mathbb{A} \sim \mathbb{N}$. Justification:

- For any $f(x) \in \mathbb{Q}[x] \backslash \{0\}$, define $Z[f(x)] = \{\zeta \in \mathbb{C} : \zeta \text{ is a root of } f(x)\}$. ($Z[f(x)]$ is the set of all roots of $f(x)$ in $\mathbb{C}$.)
  $Z[f(x)]$ is finite, and hence $Z[f(x)] \lesssim \mathbb{N}$.
  Since $\mathbb{Q}[x] \backslash \{0\}$ is countably infinite, we may 'arrange' all the elements of $\mathbb{Q}[x] \backslash \{0\}$ as an infinite sequence without repeating terms $\{f_n(x)\}_{n=0}^{\infty}$, so that $\mathbb{Q}[x] \backslash \{0\} = \{f_n(x) \mid n \in \mathbb{N}\}$. (How? Why?)
  Now $\mathbb{A} = \bigcup_{n=0}^{\infty} Z[f_n(x)]$. Then $A \lesssim \mathbb{N}$.
  Note that $\mathbb{N} \lesssim \mathbb{A}$. Therefore we have $\mathbb{A} \sim \mathbb{N}$.

Remark. It follows that $\mathbb{A} < \mathbb{C}$ and $\mathbb{A} \cap \mathbb{R} < \mathbb{R}$. There are real/complex numbers which are not algebraic; actually there are much more (real/complex) transcendental numbers than there are (real/complex) algebraic numbers. This is a non-constructive proof of the existence of transcendenal numbers.

6. $\mathbb{N}$ is the 'smallest' infinite set.
By Cantor's Theorem, we have $\mathbb{N} < \mathfrak{P}(\mathbb{N}) < \mathfrak{P}(\mathfrak{P}(\mathbb{N})) < \cdots$.

Question (1).

Is there a set of cardinality greater than each of $\mathbb{N}$, $\mathfrak{P}(\mathbb{N})$, $\mathfrak{P}(\mathfrak{P}(\mathbb{N}))$, ...?

Answer to Question (1).

Yes, one such set is the 'union' of all these sets. To make sense of this set, we need the Axiom of Substitution.

Question (2).

Is there a set of cardinality greater than $\mathbb{N}$ and less than $\mathbb{R}$?

Answer to Question (2).

Cantor believed there was no such set.

Cantor's Continuum Hypothesis.

For any set S, if $\mathbb{N} \lesssim S \lesssim \mathbb{R}$ then ($S \sim \mathbb{N}$ or $S \sim \mathbb{R}$).

So what are $\mathbb{N}$ and $\mathbb{R}$, really? Or, what is the respective nature of these two sets? This leads us to the foundation of mathematics.